SINGULAR POINTS OF FUNCTIONS WHICH SATISFY PARTIAL DIFFERENTIAL EQUATIONS OF THE ELLIPTIC TYPE.

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IN the study of the nature of isolated singular points of harmonic functions of two variables * the following theorem may well be given a fundamental place:

I. *If the harmonic function u becomes infinite for every method of approaching the isolated singular point (x_0, y_0), then u has the form*

$$C \log \sqrt{(x-x_0)^2 + (y-y_0)^2} + v(x, y), \tag{1}$$

where C is a constant and v is harmonic at (x_0, y_0).

This theorem follows at once from well known facts concerning functions of a complex variable.† It is, however, highly desirable to obtain some other proof for it in order to be able to follow out consistently the method introduced by Riemann of deducing the theory of functions of a complex variable from the theory of harmonic functions of two real variables. Such a proof I have recently found, and it turns out that it can be at once applied to large classes of partial differential equations which include Laplace's equation in two dimensions as a very special case.

The theorem thus generalized, together with some applications, forms the subject of the present paper.

* I speak of a function of the n variables $x_1, \cdots x_n$ as harmonic at the point $(a_1, \cdots, a_n)$ if throughout the neighborhood of this point it has continuous partial derivatives of the first two orders and satisfies Laplace's equation $\sum \partial^2 u / \partial x_i^2 = 0$. I speak of it as harmonic throughout a region if it is harmonic at every point of the region. By an isolated singular point of a harmonic function I understand a point at which it fails to be harmonic, although it is harmonic at every other point in the neighborhood of this point.

† Cf. *Annals of Mathematics*, Second Series, Vol. I (1899), p. 38. The proof can be given most readily by noticing that the derivative of the function of the complex variable $x + yi$ of which u is the real part is single valued in the neighborhood of the point $x_0 + y_0 i$ and can therefore be developed about this point by Laurent's theorem. Integrating this series we have a development for the function of which u is the real part from which the theorem follows without difficulty.

1. *Laplace's Equation in n Dimensions.*

We shall consider in this section a harmonic function $u(x_1, \cdots, x_n)$ of n independent variables. If $n > 2$, theorem I takes the form

II. *If the harmonic function u becomes infinite for every method of approaching the isolated singular point $(\bar{x}_1, \cdots, \bar{x}_n)$, then u has the form*

$$C\left[\sum_{k=1}^{k=n}(x_k - \bar{x}_k)^2\right]^{\frac{2-n}{2}} + v(x_1, \cdots, x_n), \tag{2}$$

where C is a constant and v is harmonic at $(\bar{x}_1, \cdots, \bar{x}_n)$.

For the sake of simplicity of exposition we will prove this theorem for the case $n = 3$, but the proof for any larger value of n,* and also for the case $n = 2$ (Theorem I) is practically identical with that here given.

We suppose, then, that $u(x, y, z)$ is harmonic throughout the neighborhood of $P_0 = (x_0, y_0, z_0)$ and becomes infinite for every method of approaching P_0. Describe a sphere S about P_0 as centre small enough so that u is harmonic everywhere within and on the surface of S except at P_0, and let $\bar{u}$ be the function harmonic at *every* point within and on S and having on S the same values as u. The function

$$F(x, y, z) = u - \bar{u} \tag{3}$$

is then harmonic at every point within and on S except at P_0, vanishes at every point on S, and becomes infinite for every method of approaching P_0.

Let $P_1 = (x_1, y_1, z_1)$ be any point within S other than P_0, and let $G(x, y, z)$, or more explicitly $G(x, y, z; x_1, y_1, z_1)$, be the Green's function which becomes infinite at this point like $1/r$ but is otherwise harmonic within and on S and vanishes on S.

Let us now consider the surface $F(x, y, z) = c_0$ where c_0 is a numerically large constant which we take to be positive or negative according as F becomes positively or negatively infinite at P_0. This surface, which we will denote by S_0, is a closed analytic surface surrounding P_0 and lying wholly within

* That all the facts concerning Laplace's equation of which we are about to make use admit of immediate generalization to space of n dimensions is well known. Cf. Kronecker: Vorlesungen über die Theorie der einfachen und der vielfachen Integrale, Leipzig, 1894. In Vorlesungen 16 and 17, proofs of many of the theorems are given.

S, and within S_0 the function F is numerically greater than c_0. Accordingly if we denote by n the interior normal to this surface, it is clear that $\partial F/\partial n$ does not change sign on the surface, being nowhere negative if F becomes positively infinite at P_0, nowhere positive if F becomes negatively infinite.

Similarly let c_1 be a large positive constant and consider the surface (which as it happens will in this case be a sphere) $G(x, y, z) = c_1$. This surface, which we will call S_1, lies within S, surrounds P_1, and on it $\partial G/\partial n$ does not change sign. We suppose the absolute values of c_0 and c_1 to be so great that S_0 and S_1 lie wholly outside of one another.

Now apply Green's theorem to the region bounded by S, S_0, and S_1. Since F and G both vanish on S this gives

$$(4) \qquad \int_{S_1}\left(F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n}\right)dS = \int_{S_0}\left(G\frac{\partial F}{\partial n} - F\frac{\partial G}{\partial n}\right)dS.$$

Accordingly, by applying the law of the mean for integrals, we have

$$(5) \quad F_{S_1}\int_{S_1}\frac{\partial G}{\partial n}\,dS - c_1\int_{S_1}\frac{\partial F}{\partial n}\,dS = G_{S_0}\int_{S_0}\frac{\partial F}{\partial n}\,dS - c_0\int_{S_0}\frac{\partial G}{\partial n}\,dS$$

where G_{S_0} and F_{S_1} denote the values which the functions G and F take on at certain points of the surfaces S_0 and S_1 respectively. The second integrals on both sides of this equation have the value zero since G and F are harmonic at P_0 and P_1 respectively. The values of the first integrals are independent of the surfaces over which we integrate provided merely that we integrate over small closed surfaces surrounding the points P_1 and P_0 respectively. By taking the first of these surfaces as a small sphere with centre at P_1 we find at once

$$(6) \qquad \int_{S_1}\frac{\partial G}{\partial n}\,dS = 4\pi.$$

If, then, we write

$$(7) \qquad \int_{S_0}\frac{\partial F}{\partial n}\,dS = a,$$

we have

$$(8) \qquad 4\pi F_{S_1} = aG_{S_0}.$$

Now let the quantities c_0 and c_1 become infinite, so that the sur-

faces S_0 and S_1 shrink down towards the points P_0 and P_1 respectively. We get as the limit of the last equation

$$F(x_1, y_1, z_1) = \frac{a}{4\pi} G(x_0, y_0, z_0) = \frac{a}{4\pi} G(x_0, y_0, z_0; x_1, y_1, z_1) = \frac{a}{4\pi} G(x_1, y_1, z_1; x_0, y_0, z_0).* \tag{9}$$

In this formula we now regard the point P_0 as fixed, but P_1 as variable. It is important to notice that a is independent of x_1, y_1, z_1, as is obvious from its definition (7). Accordingly (9) in combination with (3) and the well known formula

$$G(x_1, y_1, z_1; x_0, y_0, z_0) = [(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]^{-\frac{1}{2}} + g, \tag{10}$$

where g is a harmonic function of x_1, y_1, z_1 at P_0, gives us precisely the formula (2) for u which we wished to establish.†

Let us now consider the behavior of a harmonic function at infinity. We will assume $n > 2$.

Let

$$r^2 = x_1^2 + \cdots + x_n^2$$

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* We make use here of a fundamental theorem concerning Green's functions, the ordinary proof of which is merely the special case of the work just done in which we take for the function F the Green's function $G(x, y, z; x_0, y_0, z_0)$. Cf. Green's original Essay, § 6.

† The proof here given may easily be so modified as to avoid the use of Green's functions. For this purpose we apply Green's theorem to the two functions u and $1/r_1$, where r_1 is the distance from (x_1, y_1, z_1) to (x, y, z). We use the same region as above except that S_1 is now a small sphere having its centre at P_1; and get in place of (4)

$$\int_{S_1} \left(u \frac{\partial(1/r_1)}{\partial n} - \frac{1}{r_1} \frac{\partial u}{\partial n} \right) dS = \int_{S_0} \left(\frac{1}{r_1} \frac{\partial u}{\partial n} - u \frac{\partial(1/r_1)}{\partial n} \right) dS + \int_S \left(\frac{1}{r_1} \frac{\partial u}{\partial n} - u \frac{\partial(1/r_1)}{\partial n} \right) dS,$$

from which follows, if we let $\int_{S_1} \frac{\partial u}{\partial n} dS = a$,

$$4\pi u(x_1, y_1, z_1) = a[(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2]^{-\frac{1}{2}} + \int_S \left(\frac{1}{r_1} \frac{\partial u}{\partial n} - u \frac{\partial(1/r_1)}{\partial n} \right) dS.$$

The integral which here remains is easily shown to be a harmonic function of (x_1, y_1, z_1) throughout S.

and suppose that $u(x_1, \cdots, x_n)$ is harmonic when $r > R$, where R is some positive constant. It follows from a theorem due to Lord Kelvin * that if we let

$$x_i' = \frac{x_i}{r^2}, \quad r' = \frac{1}{r},$$

then

$$\frac{u(x_1, \cdots, x_n)}{r'^{n-2}} = \bar{u}(x_1', \cdots, x_n') \tag{11}$$

is a harmonic function of $x_1', \cdots, x_n'$ at all points for which $r' < 1/R$ except at the point $x_1' = \cdots = x_n' = 0$.

Let us now consider the different ways in which u may behave as the point $P = (x_1, \cdots, x_n)$ moves off to infinity. It may become positively infinite for some ways in which P goes to infinity, negatively infinite for others. Let us, however, suppose that this is not the case. Then there exists a constant a such that $u + a$ does not vanish at any point for which $r > R$. Accordingly by (11) the function

$$\frac{u(x_1, \cdots, x_n) + a}{r'^{n-2}}$$

is a harmonic function of $x_1', \cdots, x_n'$ which becomes infinite for every method of approaching the point $r' = 0$, and which can, therefore, by theorem II, be expressed in the form

$$\frac{b}{r'^{n-2}} + v(x_1', \cdots, x_n')$$

where v is harmonic at the point $r' = 0$. We thus get

$$u(x_1, \cdots, x_n) = (b - a) + r'^{n-2} v(x_1', \cdots, x_n'). \tag{12}$$

Accordingly, no matter how P moves off to infinity, u approaches the finite limit $b - a$. Hence the theorem

III. *The function u being harmonic when $r > R$, it either becomes both positively and negatively infinite for different ways of going to infinity, or it approaches one and the same finite limit for every method by which the point P recedes to infinity.*†

* *Liouville's Journal*, vol. 12 (1847), p. 259.

† We note in passing the following consequence of this theorem:

If for all values of $x_1, \cdots, x_n$ the function u is harmonic and $u < M$ (or if $u > m$) then u is a constant.

If in particular u vanishes at infinity, we see from (12), in which now $b = a$, that

$$r^{n-2}u(x_1, \cdots, x_n) \tag{13}$$

approaches a finite limit at infinity. Furthermore, by differentiating (12) with regard to r we see that

$$r^{n-1}\frac{\partial u}{\partial r} \tag{14}$$

approaches a finite limit at infinity. This behavior of the functions (13) and (14) is often postulated as an additional restriction on the function u, whereas we see that it follows as a consequence of the mere requirement that u vanish at infinity.

Let us now leave the consideration of the point at infinity and look again at a finite isolated singular point P_0.

IV. *If a harmonic function u of two or more variables becomes infinite for no method of approaching an isolated singular point P_0, then by suitably changing the definition of the function at this point it can be made harmonic there as well as elsewhere.*

In the case of two independent variables this theorem is suggested at once by a similar theorem for functions of a complex variable, and can be readily proved by the method suggested in the second footnote on page 455. I owe to a remark of Professor Osgood the following method by which the theorem can in all cases be established as an immediate consequence of theorems I and II.

For this purpose let us add to u the function

$$\left[\sum_1^n (x_i - \bar{x}_i)^2\right]^{\frac{2-n}{2}}$$

or in the case $n = 2$ the function

$$\log \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

We thus have a harmonic function having P_0 as an isolated singular point and becoming infinite for every method of ap-

For in this case by III u approaches a finite limit c at infinity. Let P_0 be any point and describe a sphere with P_0 as centre. The average value of u on this sphere is known to be equal to the value of u at P_0. But by taking the sphere large enough, the average value of u on the sphere can be made to differ from c by as little as we please. Therefore the value of u at P_0 must be precisely c. But P_0 was any point. Therefore u has the value c everywhere.

proaching this point. It therefore has the form (2), or if $n = 2$ the form (1). Accordingly we have

$$u = (C-1)[\Sigma(x_k - \bar{x}_k)^2]^{\frac{2-n}{2}} + v \qquad (n > 2)$$

$$u = (C-1)\log\sqrt{(x-x_0)^2 + (y-y_0)^2} + v \qquad (n = 2)$$

But since u does not become infinite as we approach P_0, we have $C = 1$ and therefore $u = v$. Accordingly u is harmonic at P_0.

I will add that precisely the same method may be applied to prove the following theorem:

V. *If a harmonic function u of two or more variables becomes infinite for some but not for all methods of approaching an isolated singular point P_0, then it can be made to become both positively and negatively infinite (and therefore also to take on every real value an infinite number of times) by suitably approaching P_0.*

2. *The Elliptic Equation with two Independent Variables.*

We will consider in this section the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0, \tag{15}$$

where a, b, and c denote real functions of the two independent real variables x, y analytic at every point of a region T of the x, y plane. We shall find it convenient to speak of a function u as being harmonic with regard to (15) at a given point, if at this point and throughout its neighborhood it has continuous first and second partial derivatives and satisfies (15). We will say that u is harmonic throughout a region with regard to (15) if it is harmonic at every point of this region with regard to (15). According to a remarkable theorem of Picard,* a function harmonic with regard to (15) at a point is analytic at that point.

In a recent dissertation † E. R. Hedrick has confirmed and made more precise a guess of Sommerfeld ‡ by proving the following theorem:

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* *Journal de l'École Polytechnique*, Cah. 60 (1890).

† "Ueber den analytischen Character der Lösungen von Differentialgleichungen," Göttingen, 1901.

‡ Encyclopädie, II A 7 c, pp. 515 and 570.

If $P_0 = (x_0, y_0)$ is any point of T, there exists a function harmonic with regard to (15) throughout the neighborhood of P_0, except at P_0 itself, and having the form

$$(16) \quad w = \log \sqrt{(x-x_0)^2 + (y-y_0)^2} \cdot U(x, y) + V(x, y)$$

where U is harmonic with regard to (15) at P_0, $U(x_0, y_0) = 1$, and V is analytic at P_0.

Now the theorem we wish to prove is this:

VI. *If the function u harmonic with regard to (15) has an isolated singular point at P_0* and if u becomes infinite for every method of approaching P_0, then u has the form*

$$(17) \quad u = Cw + v$$

where C is a constant, w is the function (16), and v is a function harmonic with regard to (15) at P_0.

To prove this theorem let us begin by considering the special case of (15) in which $c \equiv 0$. Draw a circle S about P_0 as centre and so small that not only the function w (formula (16)) is harmonic with regard to (15) within and upon S, except at P_0, but that a Green's function† $G(x, y; x_1, y_1)$ exists for every point $P_1 = (x_1, y_1)$ within S. Let $\bar{u}$ be the function which takes on the same values as u on S and is harmonic with regard to (15) throughout the interior of S, and form the function

$$F(x, y) = u - \bar{u}.$$

Let us now surround the points P_0 and P_1 (these points being supposed distinct) by small closed curves S_0 and S_1 respectively on which the functions F and G have the numerically large values c_0 and c_1 respectively; and let us suppose these values to be taken so large that S_0 and S_1 lie wholly outside of each other. Now apply the Riemann-Darboux extension of

*It must be clearly understood that we assume a, b, c to be analytic at P_0. That is we consider not the fixed isolated singular points of the differential equation, but the movable singular points of its solutions.

†That is a function of (x,y) which vanishes on S, becomes logarithmically infinite at P_1, and is harmonic with regard to the equation adjoint to (15) within and upon S except at P_1. Cf. Sommerfeld, Encyclopaedie, Vol. II, p. 570, where it is stated that the existence of this Green's function follows from Mr. Hedrick's theorem. This is in fact the case, but only after Mr. Hedrick's work has been completed by an investigation of the dependence of w (formula (16)) on the parameters (x_0, y_0); an investigation which is also necessary for the purposes which Mr. Hedrick had in view.

Green's theorem (cf. Encyclopädie, Volume II, page 513), to the region bounded by S, S_0, S_1. When we remember that throughout this region F satisfies (15) and G satisfies the adjoint of (15), while both of these functions vanish on S, we see that Green's theorem reduces to

$$\int_{S_1}\left[F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n} - (a\cos\phi + b\sin\phi)FG\right]ds$$
$$= \int_{S_0}\left[G\frac{\partial F}{\partial n} - F\frac{\partial G}{\partial n} + (a\cos\phi + b\sin\phi)FG\right]ds,$$

where n denotes the normal to the curves S_0 and S_1 which points away from the points P_0 and P_1, and ϕ denotes the angle which this normal makes with the axis of x. Using formula (9) on page 515 of Sommerfeld's article* (Encyclopaedie, Volume II) we see that the first member of this equation has the value $2\pi F(x_1, y_1)$. We have then

$$2\pi F(x_1, y_1) = \int_{S_0} G\frac{\partial F}{\partial n}ds$$
$$+ c_0\int_{S_0}\left[-\frac{\partial G}{\partial n} + (a\cos\phi + b\sin\phi)G\right]ds.$$

The last integral here is zero as we see at once by applying Green's theorem to the two functions G and 1, and to the region bounded by S_0.† Applying the law of the mean to the first integral, as we have a right to do since $\partial F/\partial n$ does not change sign on S_0, we can write

$$2\pi F(x_1, y_1) = G_{S_0}\int_{S_0}\frac{\partial F}{\partial n}ds.$$

Let us now allow the numerical value of c_0 to increase indefinitely, so that S_0 shrinks down toward the point P_0. The quantity G_{S_0} then approaches $G(x_0, y_0; x_1, y_1)$ as its limit; and, since this is different from zero,‡ we see that the integral in

* There is a mistake of sign in this formula, as is easily seen by comparing it with the familiar formula for Laplace's equation which is a special case of it.

† Our restriction up to this point to the case $c \equiv 0$ was in order to make 1 a solution of (15), this being essential for the step here taken.

‡ The equation (15) when $c \equiv 0$ shares with the special case of Laplace's equation the property that a function harmonic with regard to (15) at a certain point cannot have a maximum or a minimum there, for otherwise by

the last formula also approaches a finite limit a. Thus we have

$$F(x_1, y_1) = \frac{a}{2\pi} G(x_0, y_0; x_1, y_1) = \frac{a}{2\pi} H(x_1, y_1; x_0, y_0)$$

where H denotes the Green's function for the adjoint of (15) (cf. Sommerfeld, p. 516). But H has the form (16) and therefore F, and so also u, have the form (17), as was to be proved.

Having thus proved Theorem VI in the special case $c \equiv 0$, we get the proof in the general case by means of the following transformation.

Let $\psi(x, y)$ be a function harmonic with regard to (15) at P_0 and having a value different from zero at this point. Then, making the transformation:

$$u = \psi t,$$

we find that t satisfies an equation of the form

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + a_1 \frac{\partial t}{\partial x} + b_1 \frac{\partial t}{\partial y} = 0, \tag{18}$$

where a_1, b_1 are analytic at P_0. Now if u becomes infinite for every method of approaching P_0, t will do so too, and therefore, by what we have just proved, t becomes logarithmically infinite at P_0. From this we readily infer that u has the form (17).

A second theorem concerning equation (15) is the following:

VII. *If a function harmonic with regard to (15) becomes infinite for no method of approaching an isolated singular point, then, by suitably changing the definition of the function at this point, it can be made harmonic there as well as elsewhere.*

The proof of this theorem being practically the same as that of theorem IV need not be repeated.*

A theorem analogous to V may also be proved.

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subtracting a suitable constant, we should have a solution of (15) vanishing around a contour, which can be made as small as we please, and yet not vanishing identically, and this is well known to be impossible. Now $G(x_0, y_0; x, y)$, when regarded as a function of its last two arguments is, harmonic with regard to (15), and, since it vanishes on S and becomes negatively infinite at P_0, it could not vanish within S unless it had a maximum at some point within S.

* We note in passing that theorems VI, VII admit of immediate extension first to the case in which the second member of (15) is a given analytic function of (x, y), and second, by a change of independent variables, to the general linear differential equation with two independent variables and of the elliptic type. The statement of VII would require no modification in this general case, while the modification necessary in the statement of VI will be readily seen.

3. *More General Differential Equations.*

There seems little doubt that the results of this paper can be extended to all differential equations of the form

$$\sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{i=n} a_i \frac{\partial u}{\partial x_i} + cu = f \qquad (n > 2) \tag{18}$$

where $a_1, \cdots, a_n$, c, f are analytic functions of the n independent variables $(x_1, \cdots, x_n)$. The only difficulty here consists in the proof of the existence of a Green's function for a sufficiently small region S, which for the sake of simplicity we may take as spherical; *i. e.*, a function which, except at an arbitrarily given point P_1 in S, is continuous and has continuous first and second partial derivatives throughout S, and satisfies the adjoint equation

$$\sum_{i=1}^{i=n} \frac{\partial^2 u}{\partial x_i^2} - \sum_{i=1}^{i=n} \frac{\partial (a_i u)}{\partial x_i} + cu = 0, \tag{19}$$

and vanishes on S, while at P_1 it has the form

$$U \cdot r^{2-n} + V, \tag{20}$$

where r denotes the distance from P_1, while U and V are analytic at P_1, and U has the value 1 at this point, and satis fies (19).

Granting the existence of this Green's function, the methods already used give at once the following theorem:

VIII. *If throughout the neighborhood of a point P_0 at which the coefficients of (18) are analytic, a function u is continuous and has continuous first and second partial derivatives and satisfies (18), but if all these conditions are not satisfied at P_0, then*

(a) *if u becomes infinite for every method of approaching P_0 it has the form*

$$u = U \cdot r^{2-n} + V$$

where r is the distance from P_0, U and V are analytic at P_0, and U satisfies the equation obtained from (18) by replacing the second member by zero;

(b) *if u becomes infinite for no method of approaching P_0 it can by a suitable change of definition at this point be made continuous there, in which case it will also have continuous first and second derivatives at P_0, and will satisfy (18) there;*

(c) *if u becomes infinite for some but not for all methods of approaching P_0, it takes on every value an infinite number of times in the neighborhood of P_0.*

HARVARD UNIVERSITY, CAMBRIDGE, MASS.,
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ERRATA IN GAUSS'S "TAFEL DER ANZAHL DER CLASSEN BINÄRER QUADRATISCHER FORMEN."

BY THE LATE PROFESSOR A. M. NASH.

(Communicated by Professor E. B. ELLIOTT.)

ON page 141 of the present volume 9 of the BULLETIN Professor Hewes has exhibited the details, on Cayley's plan, of the classes of properly primitive reduced forms with the two negative determinants — 468 and — 931, which Perott has shown must be added to those given as irregular by Gauss. There is at present in my charge a mass of calculation and tabulation prepared by the late Professor A. M. Nash, of the Indian Educational Department, who died in 1895 while still engaged on the work. He seems to have possessed methods by which he could tabulate irregular determinants with few if any probable omissions. Unfortunately however his numerical work is accompanied by but little explanation of the methods and their justification. The determinants which he has tabulated as irregular cover a range up to about — 20,000, even indices, indices 3 and 9, and index 5 being separately dealt with. For about half of them the details of classes are worked out on Cayley's plan, and the composition of the principal genus given. He has also worked out in detail the classes of binary quadratic forms with all negative determinants, both regular and irregular, up to — 2,600, the results in this section of his research being marked nearly to the end "checked Gauss."

To print the whole is not feasible. The quantity alone would be prohibitive even had the principles and conclusions been left in form for publication. It is hoped however that the manuscript may be placed in some library to which mathematicians who may be desirous in the future of testing and extending the tables, or may be engaged in research on the subject, can have access. What Professor Hewes's note now suggests to me is the desirability of printing the following transcript of a list which is headed by Professor Nash "Errata in Table of Negative Determinants, Gauss, Vol. II." The list

includes many corrections not given by Schering in his appendix to Gauss's volume, or by Perott.

	For.	*Read.*
468	*Reg.*	*Irr.* 2
485	IV 4	IV 5
544	*Reg.*	*Irr.* 2
547	*Reg.*	*Irr.* 3
557	II 11	II 13
647	I 25	I 23
894	IV 6	IV 7
931	*Reg.*	*Irr.* 3
933	IV 3	IV 4
972	*Reg.*	*Irr.* 3
993	IV 4	IV 3
1116	IV 9	IV 6
1261	II 10	IV 5
1367	I 27	I 25
1396	IV 7	II 14
1508	*Reg.*	*Irr.* 2
1598	*Reg.*	*Irr.* 2
1660	IV 4	Omit
1683	II 9	II 6
1700	———	IV 12
1701	*Reg.*	*Irr.* 3
1725	*Reg.*	*Irr.* 2
1796	IV 10	II 20
1836	*Reg.*	*Irr.* 3
1872	*Reg.*	*Irr.* 2
1937	———	IV 12
1982	IV 12	Omit
1940	IV 8	IV 10

	For.	*Read.*
2331	*Reg.*	*Irr.* 2
2196	*Reg.*	*Irr.* 2
2180	*Reg.*	*Irr.* 2
2304	*Reg.*	*Irr.* 2
2320	*Reg.*	*Irr.* 2
2624	*3*	*2*
2336	*Reg.*	*Irr.* 2
2900	*Reg.*	*Irr.* 2
2188	*Reg.*	*Irr.* 3
2085	VIII 5	VIII 4
2096 in IV 6	2096	2097
2204	IV 11	IV 13
2221 in IV 9	2221	2224
2376 in IV 12	2376	2366
2448	*Irr.* 2	*Reg.*
6032	*Reg.*	*Irr.* 2
6068	*Reg.*	*Irr.* 2
6084	*Reg.*	*Irr.* 2
6148	*Reg.*	*Irr.* 2
6176	*Reg.*	*Irr.* 2
9104	*Irr.* 2	*Reg.*
9108	*Reg.*	*Irr.* 2
9156	*Reg.*	*Irr.* 2
9216	*Reg.*	*Irr.* 2
9324	*Reg.*	*Irr.* 2
9513	*Irr.* 2	*Reg.*
9554	*Reg.*	*Irr.* 2
6075	*Irr.* 3	*Irr.* 9

THE LOGARITHM AS A DIRECT FUNCTION.

BY DR. EMORY MCCLINTOCK.

(Read before the American Mathematical Society, February 28, 1903.)

In a paper of the same title published in the *Annals of Mathematics* for January, 1903, Mr. J. W. Bradshaw defines $\log x$ as a direct function of x, namely,

$$\log x = \int_1^x x^{-1} dx.$$

Attention being thus drawn to the subject, I think the time opportune to repeat and amplify a proposition of my own for the same general purpose.

In 1879 (*American Journal of Mathematics*, II, 101, etc.) I spoke of "the difficulty of comprehending logarithms," and quoted De Morgan's dictum that "the only definition of $\log x$ used in analysis is y, where $e^y = x$." After discussing this definition I said, "Another and, when duly weighed, most satisfactory definition may be derived from any one of an unlimited number of vanishing fractions, special cases of the general form $\log x = h^{-1}(x^{(1-a)h} - x^{-ah})$, where h is infinitely reduced. * * * This fraction is doubtless novel, though one case of it, where $a = 0$, is known. Even that case has not, I presume, been suggested heretofore as a definition. * * * The various theorems pertaining to logarithms may be derived with the utmost facility by the aid of these vanishing-fraction definitions. Thus, if $a = 0$, we have by expansion

$$\log(1+x) = \frac{(1+x)^h - 1}{h}\,[h=0] = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots."$$

To develop this proposition more fully, let us consider the function $y = h^{-1}(x^h - 1)$. Let k be positive, and, first, let $h = k$. When $x = 1$, $y = 0$; when $x = \infty$, $y = \infty$; and as x increases continuously from 1 towards ∞, there is one and only one corresponding value of y, which increases accordingly from 0 towards ∞. Secondly, let $h = -k$. Here again, when $x = 1$, $y = 0$; and, in the function $y = k^{-1}(1 - x^{-k})$, as x increases continuously from 1 towards ∞, there is one and only one corresponding value of y, which increases accordingly from 0 to k^{-1}, a limit which tends towards ∞ if k tends towards 0. When $h = -k$, $y = k^{-1}(x^k - 1)x^{-k}$, which differs only by the factor x^{-k} from the value of y when $h = k$. The smaller k is taken, the nearer this factor is to 1, so that the limit of the value of y, for $h = 0$, is the same whether h is positive or negative, while

$$y_{[h<0]} < \lim_{h=0} y < y_{[h>0]}.$$

The limit is therefore a 1 to 1 function of $x > 1$. When $x = 1$, the limit is 0. When $0 < x < 1$, let $x = u^{-1}$, where $u > 1$; then we have $h^{-1}(x^h - 1) = h^{-1}(1 - u^h)x^h$, and, since $\lim_{h=0} x^h = 1$, $\lim_{h=0} h^{-1}(x^h - 1) = -\lim_{h=0} h^{-1}(u^h - 1)$.

Let us define the logarithm of x (positive) as $\lim_{h=0} h^{-1}(x^h - 1)$, and denote it by $\log x$. We have just found that $\log (x^{-1}) = -\log x$. Since $\lim_{h=0} b^h = 1$,

$$\log a = \lim_{h=0} h^{-1}(a^h b^h - b^h) = \log (ab) - \log b.$$

This is the chief property of logarithms. Hence, $\log (a^2) = 2 \log a$, $\log (a^n) = n \log a$, and if $b = a^n$, lo $(b^{1/n}) = 1/n \log b$, which might be used, as by Mr. Bradshaw from another definition of $\log x$, to show that for every positive number b there exists one and only one positive nth root. Here n is a whole number. It follows that $\log b^{m/n} = m \log (b^{1/n}) = m/n \log b$. If we take n incommensurable, let $a = b^m$, where m is an integer, $b = 1 + c$, and $-1 < c < 1$. Employing the binomial expansion,

$$\log (b^n) = \lim_{h=0} h^{-1}[(1 + c)^{nh} - 1] = n(c - \tfrac{1}{2}c^2 + \tfrac{1}{3}c^3 - \cdots)$$
$$= n \log (1 + c) = n \log b.$$

Hence

$$\log (a^n) = \log (b^{mn}) = m \log (b^n) = nm \log b = n \log a.$$

That the continuous function $\log x$ has a continuous derivative x^{-1} may be shown thus, with $\Delta x < x$:

$$\frac{d \log x}{dx} = \lim_{\Delta x=0} \lim_{h=0} h^{-1}(\Delta x)^{-1}[(x + \Delta x)^h - x^h].$$

If we expand the part within the brackets and divide the resulting series throughout by $h\Delta x$, we have

$$\frac{d \log x}{dx} = \lim_{\Delta x=0} \lim_{h=0} \left[x^{h-1} + \tfrac{1}{2}(h - 1)(\Delta x)x^{h-2} + \right.$$
$$\left. \frac{1}{2 \cdot 3}(h - 1)(h - 2)(\Delta x)^2 x^{h-3} + \cdots \right].$$

If we first put $h = 0$, the part within brackets becomes $x^{-1} - \frac{1}{2}(\Delta x)x^{-2} + \frac{1}{3}(\Delta x)^2 x^{-3} - \cdots$, which is x^{-1} when $\Delta x = 0$. If we first put $\Delta x = 0$, the part within brackets becomes x^{h-1}, which is x^{-1} when $h = 0$.

THE THEORY OF AUTOMORPHIC FUNCTIONS.

Vorlesungen über die Theorie der Automorphen Functionen. Von ROBERT FRICKE und FELIX KLEIN. Leipzig, B. G. Teubner. Bd. I: *Die gruppentheoretischen Grundlagen*, 1897, xiv + 634 pp. Bd. II: *Die functionentheoretischen Ausführungen und die Anwendungen*; Erste Lieferung: *Engere Theorie der Automorphen Functionen*, 1901, 282 pp.

SIDE by side with the growth of general function theory special classes of functions have developed in which the general theories have found abundant opportunity to display their fertility and power. On the other hand, the study of special functions has repeatedly afforded the stimulus and suggested the path for new investigations along general lines. When, in addition, the intrinsic value and usefulness of such functions as the elliptic, hyperelliptic, abelian, hypergeometric, Bessel, etc., is taken into consideration, it is readily seen that the cultivation of such special fields is scarcely second in interest and importance to that of the general theory itself.

Among the various classes of special functions which have hitherto engaged the attention of mathematicians, that of most recent origin, and of by far the largest content (at least potentially) is the automorphic functions. This vast subject, the growth of the past quarter of a century, owes its rapid development to the genius and assiduity of the two eminent mathematicians, Klein and Poincaré, as well as to the comparatively high state of perfection of other mathematical disciplines which have been forced to contribute their assistance to this new field. Klein and Poincaré have each brought to this subject a breadth of knowledge and a corresponding wealth of ideas truly remarkable at the present day when multiplicity of interests hardly permits the investigator any other choice than to specialize within limits more or less narrow.

Klein in particular has by precept and example urged the importance of a closer unity among all departments of mathematical thought, and the great advantage to be derived from bringing to the aid of any one field the combined resources of all the others. It is in his work on the automorphic functions that he has given the most brilliant illustration of this mode of

treatment. In this work he has been ably seconded by the efforts of the pupils and co-laborers he has been so fortunate as to gather about him at the Göttingen school, among whom Robert Fricke is especially deserving of the highest praise for the ability and industry which he has brought to the arduous labor of preparing for publication the volumes now before us, and for the wealth of material which he himself has contributed to the subject.

These volumes form the sequel to, and final elaboration of the ideas contained in the volumes previously published, namely, the Ikosaeder (by Klein), and the Elliptische Modulfunctionen* (by Klein and Fricke). The former of these works deals exhaustively with the finite groups of transformations on a single variable, and the functions associated with them. The two large volumes of the latter treat in a very elaborate manner the modular group, and the general ideas necessary to be followed out in the study of all such groups and their functions. One of the principal objects of this fullness of discussion in the Modulfunctionen is to pave the way for a subsequent discussion of the general theory of automorphic functions without the restraint of burdensome details.

The advantage of thus disposing of preliminary details, and familiarizing the reader with many of the fundamental ideas as applied to a concrete example is clearly perceived when we observe how completely are the various elements in the present work fused into an organic whole, each part in strong and vital contact with every other part. The degree of abstractness and concentration thus attained is necessary for a forceful treatment of a subject which seeks to embrace in one view such an infinite variety and complexity of phenomena.

The theory of automorphic functions has for its object to investigate the discontinuous groups of linear transformations of a single variable ζ, and to study the properties of functions which are invariant for the transformations of any given group.

The most important aid in this work comes from geometry,† which affords a concrete representation of the values of the

* The interesting review of the Modulfunctionen by Professor Cole, and the general survery of Klein's work and ideas which it contains, renders it unnecessary for us to give more than a passing notice to the antecedents of the present work. See BULLETIN, 1st Series, vol. 1 (1892), p. 105.

† Geometry is, in fact, at the present stage of development of the subject, an indispensable tool. To the lack of geometrical aids is in no small measure to be attributed the meager success that has attended the various attempts to investigate automorphic functions of more than one variable.

complex variable ζ and pictures the effect of a group of transformations applied to this variable. In the Automorphe Functionen an introductory chapter of about sixty pages is accordingly devoted to the discussion of some geometrical notions not explained in the Modulfunctionen.

The first geometrical interpretation of the complex variable ζ is serviceable only for a restricted, but very extensive class of groups, the *fundamental circle* groups (Hauptkreisgruppen). The values of ζ are associated with the homogeneous coördinates (z_1, z_2, z_3) of points in a plane by means of the relation

$$= \zeta \frac{z_2 + \sqrt{z_2^2 - z_1 z_3}}{z_3}.$$

Collineations which leave the conic

$$z_2^2 - z_1 z_3 = 0 \tag{1}$$

unaltered correspond to linear transformations

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \tag{2}$$

on ζ. Taking the conic (1) as the absolute for a system of non-euclidean measurement of distance, the transformations (2) leave distances unaltered, and hence they are called (in a generalized sense) motions. Congruent figures, that is, figures which transform into each other have like (non-euclidean) areas. The interior of the conic, regarded for convenience as an ellipse, is called the "hyperbolic" plane.*

Out of the ∞^6 transformations of the continuous group (2) let us suppose a discontinuous group Γ selected by any suitable definition. This group is then *properly*, or *improperly* discontinuous, according as the fundamental region for Γ has a finite or an infinitesimal area. By "fundamental region" is meant a division of the hyperbolic plane such that no two points of the region are congruent, while every point without the region is congruent to some point within. The properties

* The cases in which the conic is imaginary (giving the "elliptic" plane), or breaks up into the circular points at infinity (giving the ordinary, or "parabolic" plane) lead to groups of finite order, or groups associated with the elliptic functions. These cases receive their full share of attention in the work under review, but from lack of space we pass them by without discussion.

of a given group are all implicitly involved in the geometrical properties of the fundamental region which it defines, and, accordingly, a large portion of what follows in volume I is devoted to a geometric study of the fundamental region and the network of congruent regions which arise from it by transformations of the group.

Although, evidently, the fundamental region can be selected with a high degree of arbitrariness, it can always be defined (and this in an infinity of ways) so as to be bounded by straight lines. The fundamental region is then a polygon, and the group belonging to it is called a polygon group.

A particularly useful choice for a fundamental polygon, and one which plays a leading rôle throughout the book, is determined as follows: Let $C_0, C_1, C_2, \cdots$ be any set of congruent points. Around the points C_i construct circles K_i of equal radii r, meaning by circle in this connection the locus of all points whose non-euclidean distance from a given C_i is constant. Assuming r at first sufficiently small so that no two circles collide, let the radii increase simultaneously and at the same rate for all the circles K_i with the understanding that the lengthening of any particular ray emanating from C_i shall cease the instant it meets a like ray emanating from a neighboring point C_k. Imagine this process to continue until the entire hyperbolic plane is filled without gap and without overlapping. The locus of the end points of the rays departing from a center C_i will consist of a closed chain of straight lines forming the boundary of a convex polygon. This region, called the *normal polygon*, is evidently a fundamental region for the group.*

Now let the values of ζ be associated in the ordinary way with the points of a complex plane. Then, assuming $z_2^2 - z_1 z_3$ to be negative for the interior of the ellipse, to each point of the hyperbolic plane correspond two points of the ζ-plane representing conjugate imaginary values of ζ. To each point on the ellipse corresponds one point on the real ζ-axis, and to

* Each of the points C_i is the center of a normal polygon. This polygon clearly contains within itself all points which are nearer to C_i (in the non-euclidean sense) than to any congruent point. Each polygon is equal in all respects to every congruent polygon (still speaking in the non-euclidean sense). This picturing of the group with a network of equal polygons filling up the hyperbolic plane, is seen to be a generalization of the idea of a network of equal parallelograms filling the ordinary (or parabolic) plane, with which the theory of elliptic functions has made us familiar.

each point outside the ellipse corresponds a pair of points on the real axis. On account of the one-to-two correspondence it is evidently sufficient to take into consideration only half of the ζ-plane. Moreover, by a linear transformation of ζ the real axis changes into a circle C the interior of which has a one-to-one correspondence to the half ζ-plane and hence to the hyperbolic plane.

In the hyperbolic plane the polygons crowd together in infinite number along the ellipse which forms its boundary, since this is the infinite element in the plane. Any set of congruent points, such as the C_i above mentioned, will have limiting points, or points of accumulation, on this boundary. These limit points may cover the ellipse everywhere densely, in which case the ellipse, and correspondingly the circumference of the circle C, forms a natural boundary for the group and its network of polygons.* The group is then called a *limit circle group* (Grenzkreisgruppe). On the other hand, segments of the ellipse (or circle C) of finite length may be free from limiting points. These two classes, which include all groups whose fundamental regions fill the hyperbolic plane, are specified by the general term, *fundamental circle groups*.

The foregoing method of associating the ζ values with the real points of the hyperbolic plane is of practical value only when the substitutions of the group have real coefficients since it is only in that case that real points in the hyperbolic plane are transformed into real points.* This difficulty is obviated by a more general procedure. A quadric surface is taken in ordinary space, for convenience the sphere

$$z_1^2 + z_2^2 + z_3^2 - z_4^2 = 0. \tag{3}$$

The values of ζ are associated with points on the surface of the sphere in the familiar manner of the Riemann function theory by means of the relation

* It follows that any function, invariant for transformations of the group, has the circle for natural boundary and cannot be continued over this boundary which is everywhere filled with essential singularities of the function.

* That there is a practical necessity for the restriction to real coefficients is not mentioned by our authors, although they may have expected it to be inferred. This inference, however, is not likely to be at once drawn by the reader, since the reason assigned in the text is that only in this case are groups of "motion," or motion combined with inversion, obtained, without explaining why it is desired to exclude from consideration all groups other than these.

$$\zeta = \frac{z_1 + iz_2}{z_4 - z_3}. \tag{4}$$

Every collineation which transforms the sphere into itself subjects ζ to a linear transformation of one of two kinds,

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \qquad \zeta' = \frac{\alpha\bar{\zeta} + \beta}{\gamma\zeta + \delta}, \tag{5}$$

when $\bar{\zeta}$ is the conjugate of ζ. Taking the surface of the sphere as the absolute for a system of non-euclidean measurement in space, the interior of the sphere is called the hyperbolic space. The non-euclidean distance between two points is unchanged by the transformations (5). Those of the first type are "movements" of the hyperbolic space, while one of the second type is a reflection, or symmetrical transformation with respect to a certain plane of symmetry.

The values of $\zeta = \xi + i\eta$ are likewise represented by the coördinates of points in a complex plane referred to ξ, η rectangular axes. A third rectangular coördinate ϑ in space is introduced. Then assuming

$$\vartheta = \pm \frac{\sqrt{z_4^2 - z_1^2 - z_2^2 - z_3^2}}{z_4 - z_3},$$

this equation combined with (4) establishes a correspondence between the ξ, η, ϑ space and the hyperbolic space such that to a point within the sphere correspond two points situated symmetrically with respect to the ζ plane. The portion of the ξ, η, ϑ space situated on one side of the ζ plane has a one-to-one correspondence with the hyperbolic space. This is called the ζ half-space. To each plane of the hyperbolic space corresponds a half sphere of the ζ half-space which is orthogonal to the ζ plane.

Consider now a discontinuous group Γ of transformations (5). A fundamental region for Γ will consist of a limited portion of the hyperbolic space (or of the ζ half-space) of largest possible extent subject to the condition that no two congruent points are within this region. Such a fundamental region can be selected in an infinity of ways so as to be bounded by planes alone. If this fundamental region be transformed by all the substitutions of the group, the totality of congruent regions

thus obtained will fill the hyperbolic space without gap and without overlapping. These polyhedra may or may not intersect the surface of the sphere. In the former case the surface is divided up into a network of polygons forming fundamental regions for the ζ transformations, which, by the mediation of the above correspondence (4), may be depicted on the ζ plane, while at the same time the network of polyhedra are represented by polyhedra in the ζ half-space whose faces are half spheres orthogonal to the ζ plane. The group is then properly discontinuous in the ζ plane (or on the ζ sphere).

If, in particular, the group consists of all the transformations which leave unchanged a point P outside the sphere, the polar plane of P and its circle of intersection with respect to the sphere is invariant for the group, which is accordingly a fundamental circle group. From this point of view it is called a *hyperbolic rotation group*. If the point P is on the sphere, we have the *parabolic rotation groups* with one limit point, to which the elliptic functions are related. Finally, when P is within the sphere, the polar plane does not intersect the surface in real points, and hence the group has no limit points. There are accordingly only a finite number of fundamental regions, and the group is finite. It is called an *elliptic rotation group*.

It is found that the use of the polyhedra in the hyperbolic space is an especial convenience and simplification in the study of properties of non-rotation groups, and indispensable for the investigation of groups which are improperly discontinuous on the surface of the sphere (or in the ζ plane). An example of the latter kind to which considerable space is allotted is the Picard group which consists of all the substitutions of ζ formed with complex integer coefficients whose determinant is 1 or i.

Returning to the polyhedra into which the hyperbolic space has been divided by the transformations of the group, it is further noticed that these can be determined in a manner exactly analogous to the determination of the normal polygons in the hyperbolic plane. A set of congruent points $C_0, C_1, \cdots$ is selected at random and small spheres (in the non-euclidean sense) of equal radii r described about each. Then r is imagined to increase simultaneously for all these spheres, with the understanding that any ray emanating from C_i stops increasing when it meets a ray proceeding from a neighboring center. When this process is carried to an end, the hyperbolic space will be filled without gaps and without overlapping by poly-

hedra, all of whose faces are planes. A polyhedron determined in this way is called a *normal polyhedron*. The study of the normal fundamental polyhedra here plays the same rôle in investigating the properties of the group as the normal polygon does in the case of the fundamental circle groups.

Particular attention is devoted to those non-rotation groups whose normal polyhedra intersect the surface of the sphere in one or more distinct networks of polygons. A classification of groups is based on the nature of these nets and their limit points.

There may be a single net completely covering the sphere. The limit points are then isolated, and if there are more than two of them, the number is infinite. On the other hand, the number of nets may be two, or infinite. In either case the number of limit points is infinite since the nets are separated from each other by these points. Hence non-rotation groups may be classified into:* (*a*) those with two limit points; (*b*) those with an infinity of limit points. Division *b* is classified still further according as the number of nets is (1) *one*, the limit points being isolated; (2) *two*, the limit points forming a non-analytic curve separating the two nets; (3) *infinite*, the infinity of nets being separated from each other by an infinity of limit curves formed by limit points of the group. This last case is further subdivided according as the limit curves are (at least in part †) non-analytic, or consist entirely of circles, and still further as the nets are simply or multiply (always with an infinite multiplicity) connected.

A kind of converse question naturally arises, namely, in how far can an arbitrarily chosen polygon serve as the fundamental region for a group of linear transformations, and thus serve to define and generate the group? The restrictions under which a polygon can be chosen are determined for groups in the hyperbolic plane (fundamental circle groups). The like problem is then solved for polyhedra in the hyperbolic space and the results made useful for the determination of polygons suitable for defining non-rotation groups by means of the theorem: any polygon P_0 (on the surface of the sphere) is a fundamental

*These are divisions II and IV in the book. See pp. 164–5. Division I comprises the ordinary cyclic groups formed by the repetition of a single operation, and division III comprises the rotation groups already noticed above.

† If analytic curves occur, they must be circles.

region for some discontinuous group of transformations provided that a polyhedron can be constructed intersecting the sphere in P_0 and satisfying the restrictions which polyhedra are subject to in order to be usable for fundamental regions in the hyperbolic space.

A new kind of fundamental region, the *canonical polygon*, is next introduced which greatly facilitates the discussion of the generating operations of the group and the relations existing among them. To make clear the idea, it is observed that the sides of a fundamental polygon must be congruent in pairs, and the vertices in cycles. If the sides of a fundamental polygon be deformed so as to bring together and unite congruent lines and points, a closed surface F is obtained of a certain genus p which is called the genus of the group. The surface F may be conveniently thought of as an ordinary Riemann surface.

Consider now the reverse process. Take any Riemann surface and draw from an arbitrary point E a canonical system of cross-cuts which reduces the surface to a simply connected one. It may then be deformed into a polygon suitable for the fundamental region of a group. The opposite edges of the cross-cuts become congruent edges of the polygon, and the substitution which brings an edge into coincidence with a congruent edge is one of the generators of the group. If the fundamental polygon is assumed to have vertices which are fixed points for parabolic or elliptic substitutions, then a certain number of points $e_1, e_2, \cdots, e_n$ are selected in the Riemann surface and cuts drawn from E to each of these points. If e_i is to be a fixed point of an elliptic substitution of finite period, then the deformation must be carried out so that in the end the angle at e_i is an aliquot part of π. If, on the other hand, e_i is to be a fixed point for a parabolic substitution, the angle must be zero after the deformation. The polygon so obtained is the canonical polygon. It has $2n + 8p$ edges which are congruent in pairs. It is possible to unite some of the congruent edges by a suitable deformation of the polygon. The $n + 4p$ substitutions which transform congruent edges into each other are generators of the group. Certain relations exist among these which readily permit their number to be reduced to $n + 2p$ among which a single relation exists (in addition to the relations of the form $V^l = 1$ for the elliptic substitutions).

The second division of volume I is devoted to an application of the geometrical foundation principles, as laid down in the

first division, to the detailed study of polygon groups. The normal and canonical polygons are made the leading instruments of investigation, and the cases of the elliptic and parabolic rotation groups, the non-rotation groups with two limit points, the hyperbolic rotation groups, and the non-rotation groups with an infinity of limit points are considered in turn. The first three cases are comparatively simple and easily disposed of. The fourth case is treated at great length. The important principle is first deduced that for any group Γ the normal polygon with a given center C is the region common to the normal polygons (with the same center) of all the groups whose composition generates Γ. This principle is applied in particular to the cyclic subgroups of Γ.

Various properties of the sides and corners are next considered. It is shown for example that, while at an accidental corner (that is, one which is not a fixed point for a substitution of the group) in general only three polygons meet, for special positions of the center C* more than three polygons meet in such a point. All polygons having the same genus p and number n of fixed vertices are said to be of the same kind and of character (p, n). Polygons of the same kind having a like number of sides, the sides being congruent in the same order, are said to be of the same *type*. The type is *ordinary* or *special* according as the accidental corners occur in cycles of three and the fixed corners in cycles of one, or not. The relations which are found to exist between corners, sides, and number of cycles show that the occurrence of a special type involves a reduction of the number of sides s. Special types with $s - 4$ sides occur with each group and arise from a particular choice of the center C. Special types in which the number of sides is diminished by six or more below the number belonging to the ordinary type can occur only for special groups. There are called *singular groups*.†

*In fact C will lie either on the ellipse, or on a certain curve of the third degree.

†Our authors do not attempt to develop a theory of singular groups, but content themselves with remarking on the possible importance of these groups in relation to the theory of algebraic functions. The reader of the Automorphe Functionen will have frequent occasion, as here, to notice the occurrence of gaps in the development of the subject. While the work under review may rightly be regarded as presenting a systematic and well-developed theory of the automorphic functions, a vast amount of work yet remains to be done. To mention only one further instance, it may be remarked that the theory of non-rotation groups has not yet been systematized, and in spite

The second chapter in this division of the book deals with the canonical polygon with special reference to the form it may take in the hyperbolic plane, and the transformations it may undergo on account of transformation from one canonical system of cross-cuts in the Riemann surface to any other. The details are first worked out for polygons of character (0, 3), (1, 1), and (0, n). The methods being made clear by these simplest cases, the case (p, n) is then treated in its generality. The remainder of the chapter is devoted to an extended discussion of the *moduli* (Moduln). The moduli are the parameters on which the substitution coefficients depend, and which are invariant when the group is transformed by any linear substitution. The groups so obtained are said to belong to the same class, and the moduli form a system of numbers characteristic for that class. As the groups of the same class have like structure, a determination of the properties of the class acquaints us with the properties of any group of the class. Any particular set of possible values for the moduli determines a class, and the number manifoldness of all the classes is measured by that of the moduli.

The third and last division of volume I treats of the arithmetic definition of discontinuous groups. This subject, although of fundamental importance, offers at the present time only fragmentary results on account of the great difficulties that it has to overcome.

The first chapter determines the arithmetic characteristics of the rotation subgroups of the Picard group. This is accomplished by the introduction of the Dirichlet and the Hermite quadratic forms, concerning which the following two theorems are proved: Every hyperbolic or loxodromic cyclic subgroup transforms into itself a particular Dirichlet form having for determinant a non-square; and, conversely, every such form is invariant for some such subgroup. Again, every fundamental circle group contained within the Picard group transforms into itself a particular indefinite Hermite form; and conversely, every such form is reproduced by some fundamental circle subgroup of the Picard group.

The second chapter discusses the groups that reproduce cer-

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of the considerable space devoted to this subject and the powerful aid afforded by the introduction of the fundamental polyhedron in the hyperbolic space, this portion of the field can hardly be said to have been more than lightly touched upon. (Compare the remarks on page 441, Vol. I.)

tain ternary and quaternary forms. This includes a large and important class of groups, among which the Picard group occurs as the reproducing group of a particular quaternary form. A considerable number of illustrative cases are worked out.

The third chapter deals with groups whose substitutions have coefficients which are integers within a given field of algebraic numbers.

Volume II, of which the first part only has appeared, is devoted to a study of the automorphic functions belonging to the groups treated in the first volume. By an automorphic function of an independent variable ζ is meant a function $\phi(\zeta)$ which is unaltered when ζ is transformed by any substitution

$$\zeta' = \frac{\alpha_k \zeta + \beta_k}{\gamma_k \zeta + \delta_k} \tag{6}$$

of a given group Γ; that is,

$$\phi\left(\frac{\alpha_k \zeta + \beta_k}{\gamma_k \zeta + \delta_k}\right) = \phi(\zeta).$$

It is further required that $\phi(\zeta)$ have no essential singularity at any point of the fundamental region of Γ, and that within this region it is uniform and without branching. Accordingly $\phi(\zeta)$ is expansible in the vicinity of an ordinary point ζ_0 in powers of $\zeta - \zeta_0$, and in powers of $1/\zeta$ in the vicinity of the infinite point.

If ζ_0 and ζ_0' are the fixed points of an elliptic substitution of period l, then since the corresponding substitution can be written in the form

$$\frac{\zeta' - \zeta_0}{\zeta' - \zeta_0'} = e^{\frac{2\pi i}{l}} \frac{\zeta - \zeta_0}{\zeta - \zeta_0'},$$

the function $\phi(\zeta)$ can be expanded in the vicinity of ζ_0 in a series of ascending powers of

$$\left(\frac{\zeta - \zeta_0}{\zeta - \zeta_0'}\right)^l.$$

Every parabolic substitution can be written in the form

$$\frac{1}{\zeta' - \zeta_0} = \frac{1}{\zeta - \zeta_0} + \gamma,$$

in which ζ_0 is the fixed point. In the vicinity of this point $\phi(\zeta)$ can therefore be expanded in a series of ascending powers of $e^{\frac{2\pi i}{\gamma}\cdot\frac{1}{\zeta-\zeta_0}}$.

The first step in the development of the theory of automorphic functions is to prove that such functions exist for every group Γ which is properly discontinuous in the ζ-plane. This is accomplished by means of Riemann's general existence theorem, the details of the proof following the methods of Schwarz and Neumann.

If $\zeta = \xi + i\eta$, it is first shown that an automorphic potential $u(\xi, \eta)$ can be found which satisfies prescribed boundary conditions, and is uniform and everywhere continuous in the fundamental region of Γ except at one prescribed point ζ_0 where it becomes discontinuous like the real part of $1/(\zeta - \zeta_0)$.

The conjugate potential v is defined by the integral

$$v(\zeta, \eta) = \int_{(\xi_0, \eta_0)}^{(\xi, \eta)} \left(\frac{\partial u}{\partial \xi} d\eta - \frac{\partial u}{\partial \eta} d\xi \right).$$

It is shown that dv has the automorphic character, and that hence v is reproduced by any substitution of the group with the addition of a constant. This constant is zero for groups of genus zero, and in that case the function

$$\phi(\zeta) = u(\xi, \eta) + iv(\xi, \eta)$$

is an automorphic function of ζ.

If $p > 0$, the expression $Z = u + iv$ behaves on the closed Riemann surface F, on which the fundamental region of Γ is depicted, like an elementary integral of the second kind. Hence, according to the well known Riemann method, if $Z_1, Z_2, \cdots, Z_\mu$ be μ such functions having different poles, constants $C_1, C_2, \cdots, C_\mu$ can be determined so that

$$C_1 Z_1 + C_2 Z_2 + \cdots + C_\mu Z_\mu$$

is reproduced unchanged when continued over any closed path in F. As a closed path in F corresponds to a path joining two congruent points in the ζ-plane, this means that $\Sigma C_i Z_i$ is automorphic for the group Γ.

Among the properties of $\phi(\zeta)$ may be mentioned the following. It takes any given value at μ points of the fundamental

region P, μ being the number of poles of $\phi(\zeta)$. From this follows that the region P can be conformally represented on an ordinary μ-leaved Riemann surface F of genus p by means of the function $z = \phi(\zeta)$. Any automorphic function of ζ is accordingly an algebraic function on the Riemann surface F. On the other hand every algebraic function of z, $W[z(\zeta)]$, is an automorphic function of ζ, so that the totality of automorphic functions belonging to the group Γ coincides with the totality of algebraic functions on the surface F. It follows from this that between any two functions ϕ_1, ϕ_2 which are automorphic for the group Γ an algebraic relation exists, $G(\phi_1, \phi_2) = 0$. If this relation is irreducible, then every automorphic function belonging to the same group is rationally expressible in terms of ϕ_1 and ϕ_2. In particular, if $p = 0$, every such function is rationally expressible in terms of any function having a single pole in the fundamental region. Such a function is called a *principal function*. In general, a principal function is one which takes a given value the least possible number of times within the fundamental region.

It is evident from what precedes that every algebraic function on the Riemann surface F which is in general a many valued function of z, is single valued when expressed in terms of ζ. Every Riemann surface can be associated with a group Γ (in fact with more than one group), so that every algebraic function can be expressed as a uniform function. In the language of geometry, the coördinates of a point on any algebraic curve can be expressed as uniform (automorphic) functions of a variable parameter. To establish this result is one of the main problems of the book, or more explicitly stated: given any Riemann surface of genus p and n arbitrarily assigned points on it, $e_1, e_2, \cdots, e_n$, to investigate the existence of a function $\zeta(z)$ which branches at the given points and which can mediate the representation of the Riemann surface on a polygon suitable for the fundamental region of a limit circle group and having vertices corresponding to e_i which are fixed points for substitutions of periods $l_1, l_2, \cdots, l_n$, these being any integers > 1, or infinite.

Since $\phi(\zeta)$ takes any given value μ times in every polygon which is congruent to the fundamental region P, and since these polygons crowd together in infinite number about the limit points of the group, it follows that $\phi(\zeta)$ takes any given value an infinite number of times in the immediate vicinity of

such a point which is accordingly a point of essential singularity for the function. If these limit points fill a curve, the limit curve, everywhere dense, this curve forms a natural boundary for the function over which it cannot be analytically continued.

Corresponding to the classification of the discontinuous groups, an analogous classification of the automorphic functions is given.

I. *Cyclic functions*, belonging to cyclic groups.

If the group is elliptic of period l, the principal function is the algebraic function $[(\zeta-\zeta_0)/(\zeta-\zeta_0')]^l$. If the group is parabolic, the principal function is $e^{\frac{2\pi i}{\gamma}\cdot\frac{1}{\zeta-\zeta_0}}$. If hyperbolic, or loxodromic, the automorphic functions are the elliptic functions of $\log[(\zeta-\zeta_0)/(\zeta-\zeta_0')]$.

II. *Elliptic functions.* The corresponding groups are the parabolic rotation groups.

III. *Fundamental circle functions.* These belong to groups whose limit points lie on a circle. If the circle is everywhere dense with fixed points, we have the special, but highly important, class of functions existing only within the circle and having the circumference for a natural boundary.

IV. *Automorphic functions in general, without fundamental circle.* This last and most extensive division of functions does not at present admit of precise classification owing to lack of knowledge of the non-rotation groups to which they belong. Three main subclasses are given, however, corresponding to the subdivisions of the groups mentioned above.

The inverse problem of determining the nature of ζ as a function of z in the Riemann surface F is next considered. When z describes a closed path in F, ζ is either unchanged or undergoes a linear transformation. It is infinitely many valued, and is accordingly called a polymorphic function. The important question arises as to whether in case any Riemann surface whatever is given, a polymorphic function exists, having the properties of the function ζ and capable of representing the Riemann surface on a polygon (or other region) suitable as the fundamental region for a group of linear substitutions. The full discussion of this problem is reserved for the remaining part of the volume which is not yet published, the author delaying on this point only to show that the polymorphic function ζ satisfies a differential equation of the third

order obtained by equating the "Schwarzian derivative" of ζ to a certain function which is algebraic on the surface F.

From a consideration of automorphic functions of ζ we pass on in the second chapter to the consideration of forms belonging to groups of genus zero. The variable ζ is separated into the quotient of two variables ζ_1, ζ_2. Instead of the group Γ of substitutions of the form (6), the corresponding group of homogeneous substitutions

$$(7) \qquad \zeta_1' = \alpha_k\zeta_1 + \beta_k\zeta_2, \qquad \zeta_2' = \gamma_k\zeta_1 + \delta_k\zeta_2$$

is introduced. The course of the investigation is directed towards the construction of binary forms in ζ_1, ζ_2, having the automorphic character. This treatment of the problem from the point of view of binary forms is characteristic of the methods of Klein as previously employed in the Icosahedron theory and the Modular Functions.

An automorphic form is defined as a homogeneous, non-branching function of ζ_1, ξ_2 of dimension d (d being any rational number) which is reproduced multiplied by a constant μ when the variables describe any continuous series of admissible value such that the end values ζ_1', ζ_2' are congruent to the initial values with respect to the homogeneous group of unimodular substitutions (7).

That is,

$$\phi(\alpha_k\zeta_1 + \beta_k\zeta_2, \gamma_k\zeta_1 + \delta_k\zeta_2) = \mu_k\phi(\zeta_1, \zeta_2).$$

The quotient of two forms of like dimensions and the same multipliers is evidently an automorphic function of $\zeta_1/\zeta_2 = \zeta$.

Among the various forms that could be constructed for a given group ($p = 0$) two are of especial interest, the *principal form*, and the *prime form*. The former is defined by the equation

$$\phi_{-2}(\zeta_1, \zeta_2) = -\frac{1}{\zeta_2^2}\frac{dz}{d\zeta},$$

in which z is a principal function, that is, an automorphic function of ζ which takes a given value but once in the fundamental region. The principal form is absolutely invariant, is of dimension -2, has one pole of order 2 at the point where $z = \infty$, and vanishes of order $1 - 1/l_k$ at a vertex ϵ_k of the

fundamental region which is a fixed point for a substitution of period l_k. It has no other poles or zeros.

If e_k is the value of z at $\zeta = \epsilon_k$, then $(z - e_k)$ vanishes of the first order at $\zeta = \epsilon_k$ and at all equivalent points: Hence the product

$$\phi_{-2} \prod_{k=1}^{n} (z - e_k)^{-\left(1-\frac{1}{l_k}\right)}, \tag{8}$$

in which ϕ_{-2} is the principal form, does not vanish at the vertices, and is zero of order

$$\sum_k \left(1 - \frac{1}{l_k}\right) - 2 = -\frac{2}{\nu}$$

at the pole ζ_0 of ϕ_{-2}. Accordingly, if the expression (8) is raised to the power $\left(-\frac{\nu}{2}\right)$, the result vanishes of the first order at ζ_0. Such a form, denoted by $z_2(\zeta_1, \zeta_2)$, is of dimension ν in ζ_1, ζ_2, and is finite and different from zero at every point of the fundamental region except ζ_0. It is called a prime form.

The form

$$z_1(\zeta_1, \zeta_2) = z(\zeta) \cdot z_2(\zeta_1, \zeta_2)$$

is likewise a prime form whose zero point in the fundamental region coincides with that of z.

The formula $az_1 + bz_2$ defines a binary family of prime forms, all of which behave like z_2 with respect to the substitutions of the homogeneous group, and whose zero point varies with the parameter a/b.

If (z, e_k) denote the particular form of this family which vanishes at the fixed corner ϵ_k of the fundamental region, then the form

$$Z_k(\zeta_1, \zeta_2) = \sqrt[l_k]{(z, \epsilon_k)}$$

is a non-branching form for the group. It is called the *ground form* belonging to the corner ϵ_k. In case of an elliptic point l_k is the period of the substitution, and for a parabolic point l_k may be any positive integer.

The effect of the substitutions of the homogeneous group upon any form ϕ_d is next determined. This is accomplished by calculating the multipliers μ_k which arise from the generat-

ing substitutions. From this set of multipliers, called a multiplier system, the multiplier for any other operation of the group is obtained by a very simple formula.

The number of theoretically possible multiplier systems having been determined, it is then proved that automorphic forms of any allowable dimension d exist for every such system. This is done by first showing that every automorphic form can be expressed as a product of prime forms and ground forms, and that this product can be so chosen that the system of multipliers will coincide with any given possible system. In particular, the forms with the multiplier system $\mu_k = 1$ exist for every group Γ and for every integer dimension d. They are absolutely unchanged by substitutions of the group, and for this reason are called *proper* automorphic forms.

The variables ζ_1, ζ_2 when regarded as depending on z_1, z_2 are polymorphic forms in the latter. On multiplying ζ_1 and ζ_2 by certain forms in z_1, z_2 polymorphic forms are obtained which are of zero dimension and behave like ζ_1, ζ_2 with respect to substitutions of the group. Analytic expressions for these various forms are obtained, and differential equations of the second order which they satisfy are deduced.

Some of the foregoing results are illustrated by means of the hypergeometric function.

The third chapter treats of the Poincaré series with a detailed consideration of the case $p = 0$. This subject, which is one of the most important and fundamental in the entire theory of automorphic functions, is handled in a very felicitous and attractive manner.

"Poincaré series" is the name given in the present work to the series

$$(9) \qquad \phi(\zeta_1, \zeta_2) = \sum_k \mu_k^{-1} H(\alpha_k\zeta_1 + \beta_k\zeta_2, \gamma_k\zeta_1 + \delta_k\zeta_2),$$

in which $H(\zeta_1, \zeta_2)$ is any rational homogeneous function of ζ_1, ζ_2, and μ_k is the multiplier, of any allowable system, corresponding to the transformation (7). The function ϕ thus defined is an automorphic form for the given group with the multiplier system μ_k.

If the above series be divided by ζ_2^d, d being the dimension of ϕ, and the multiplier system $\mu = 1$ taken, the resulting formula

$$\Theta(\zeta) = \sum_k H\left(\frac{\alpha_k\zeta + \beta_k}{\gamma_k\zeta + \delta_k}\right)(\gamma_k\zeta + \delta_k)^d,$$

with d a negative even integer, is the "theta" series introduced by Poincaré, and is the principal element out of which he constructs the analytic theory of the automorphic functions. This function is reproduced, multiplied by a factor, by any substitution of the group; that is,

$$\Theta\left(\frac{\alpha_i\zeta+\beta_i}{\gamma_i\zeta+\delta_i}\right)=(\gamma_i\zeta+\delta_i)^{-d}\Theta(\zeta),$$

while the corresponding form ϕ is entirely unchanged.

The series (9) is proved by two methods, both due to Poincaré, to be convergent. The first proof establishes the convergence for a group with one limit curve, or an infinite number of them, and for all dimensions $d \leqq -4$. This is the most general case and includes all others.

Under what conditions the Poincaré series is convergent for values of d greater than -4 is a problem not yet completely solved, but the case $d=-2$ receives somewhat extensive consideration. It is shown that for this value of d the series is no longer absolutely convergent whenever the group has one or more limit curves; but that for fundamental circle groups which are not limit circle groups, and for certain other groups without limit curves, the series is unconditionally and uniformly convergent. These results, which are largely due to the investigations of Burnside and Schottky, lead the authors to suggest with Burnside that the theorem is possibly true for all groups without limit curves. It is worthy of remark, also, that groups exist (for every genus p) for which the series of dimension $d=-1$ is absolutely convergent.

The second proof is given for the case of a fundamental circle group, and establishes the convergence of the series (9) for all dimensions d less than -2.

The quotient of two Poincaré series of like dimension d and with the same multiplier system is an automorphic function of $\zeta=\zeta_1/\zeta_2$. Thus the Poincaré series affords a simple and elegant formula for the construction of functions belonging to the group.

One serious obstacle in the way of using this series arises from the possibility of its identical vanishing. That this possibility is not an imaginary one is shown by proving that identically vanishing series exist, and in infinite number for every dimension d. The difficulty is avoided by the construction of

series having one or more poles in the fundamental region. This is effected by assigning poles to the function H which is used in generating the series. The case of a single pole in the fundamental region being the simplest one is considered in detail. It is proved that such a series can always be constructed, and the details of the method, and the actual form of the series are given except in the case of absolute automorphic forms of dimension -2 which must have at least two poles. It is shown that the pole can be taken at any point in the polygon net, the parabolic fixed points alone excepted.

The one-pole series is now placed in the foreground. On the introduction of a suitable multiplying constant, it becomes discontinuous at the pole $\zeta_1 = \xi_1, \zeta_2 = \xi_2$ like $1/(\zeta_1\xi_2 - \zeta_2\xi_1)$. The series so normalized is called an *elementary form* and is denoted by $\Omega(\zeta_1, \zeta_2; \xi_1, \xi_2)$. It is worthy of notice that this elementary form, when regarded as depending on ξ_1, ξ_2, is of dimension $-d-2$, and in its manner of becoming discontinuous behaves like an automorphic form with inverse multiplier system μ_k^{-1}. In order that Ω may be an automorphic form in ξ_1, ξ_2 it is necessary that, when expressed as a product of prime forms and ground forms, it be of dimension -1 in the former. It also has the same property when regarded as a form in ζ_1, ζ_2, which fact leads to the expression of Ω in this case by a very elegant formula [(12), page 201].

Except for the special case just mentioned, Ω is not automorphic in ξ_1, ξ_2. It has, however, certain properties as a function of these variables, which lead to the important conclusion that every automorphic form which vanishes in the parabolic points can be expressed in the form of a Poincaré series provided the dimension d is one for which that series is convergent. That automorphic forms exist which are not expressible as Poincaré series is shown by the occurrence of such a case for $d = -1$ in the modular group.

This result is further generalized by showing that every automorphic form with arbitrary poles none of which occur at a parabolic point, is expressible as a Poincaré series plus an integral automorphic form.

The fourth and last chapter generalizes the results to the case $p > 0$. As in this case the automorphic functions are no longer expressible in terms of a principal function z, the simple prime form for $p = 0$ must be replaced by the transcendental prime form which enters into the theory of abelian functions.

The Ritter prime form introduced here, while not so simple an analytic expression as the Klein prime form, has the advantage over the latter of not vanishing at the branch points of the Riemann surface F.

Klein's prime form $\Omega(x, y)$ is defined as the limit of the expression

$$\sqrt{-(x, dx)\,(y, dy)\,e^{-\Pi_{x,y}^{x+dx,\,y+dy}}}$$

for $dx = 0$, $dy = 0$. Here $\Pi_{x,y}^{\xi,\eta}$ is the normal integral of the third kind on the m-leaved Riemann surface, while (x, dx) is written for brevity to represent the homogeneous differential expression $x_1dx_2 - x_2dx_1$.

Ritter's prime form $P(z, e)$ is

$$\Omega(z, e)\sqrt[m]{\frac{z_2}{\Omega(z, \infty_1)\,\Omega(z, \infty_2)\cdots\Omega(z, \infty_m)}}$$

in which e is any point of the Riemann surface and $\Omega(z, \infty_i)$ is Klein's prime form with the path of integration extending to the point at infinity in the ith sheet of the surface, while z_2 is one of the homogeneous variables into which $z = z_1/z_2$ is separated. The prime form thus defined is everywhere continuous and has only a single zero of the first order at $z = e$. It is, however, in the periodic property of this prime form with respect to substitutions of the group Γ that its superiority over the Klein form is especially marked, for while the latter is reproduced multiplied by a function of z, the former has a multiplier independent of z.

Expressions are next obtained in terms of the Ritter prime form for the polymorphic forms ζ_1, ζ_2 whose quotient gives the variable ζ in terms of which z is an automorphic function.

Just as in the case $p = 0$, the variable ζ as a function of z satisfies a differential equation of the third order obtained by equating the Schwarzian derivative to a certain function R which is rational and algebraic on the Riemann surface, and which depends on $n + 3p - 3$ arbitrary constants, the accessory parameters of the differential equation. The form of the function R is determined for three cases of special interest, viz:

1. When the group Γ is of character $(1, n)$.
2. When it is hyperelliptic of character $(p, 0)$.
3. For the general case of character $(3, 0)$.

The polymorphic forms ζ_1, ζ_2 satisfy the differential equation of the second order

$$\frac{d^2F}{dx^2} + RF = 0,$$

in which R is the function occurring in the differential equation of the third order for ζ.

The general multiplicative forms, that is, the automorphic forms which are reproduced with multiplying constants by the operations of the group Γ, are now taken under consideration, the main result arrived at being the theorem that such forms always exist for every integer dimension d and for every theoretically possible multiplier system. Every form ϕ of this kind is expressible as a product of ground and prime forms giving the zeros of ϕ, divided by a product of prime forms giving the poles of ϕ. As in the case $p = 0$, the ground form is a properly chosen root of a prime form which vanishes at one of the fixed corners of the fundamental polygon. By prime form is here meant the product of the Ritter form by an exponential factor whose exponent is an abelian integral of the first kind on the Riemann surface, such an exponential factor being the most general expression for a multiplicative form without poles or zeros.

Among the automorphic forms belonging to a group of genus p those of dimension -2 and multiplier system 1 which are free from poles have particular interest. Denoting such a form by $\Phi_{-2}(\zeta_1, \zeta_2)$ it is observed that the integral $\int \Phi_{-2}(\zeta_1, \zeta_2)(\zeta, d\zeta)$ is of dimension zero and hence a function of ζ (and therefore of z) which is everywhere finite on the Riemann surface F. It is accordingly an abelian integral of the first kind. As there are p linearly independent integrals of the first kind it follows that the same number of linearly independent functions Φ_{-2} exist. The number of linearly independent forms ϕ of given dimension and multiplier system is $t - p + \sigma + 1$, in which t is the number of zeros of ϕ, and σ is the number of linearly independent forms Φ_{-2} which vanish in these t zero points. This is clearly only another form of the Riemann-Roch theorem.

The expression of automorphic forms by means of the Poincaré series next demands attention. It is to be remarked first of all that in order to insure the convergence of this series, the multipliers μ_k were assumed unimodular. If a form ϕ' has

multipliers which are not unimodular, it can be expressed in the form

$$\phi' = e^{\Sigma c_i w_i}\phi,$$

in which ϕ is a form with unimodular multipliers and the w_i are normal integrals of the first kind on the Riemann surface F. A generator of Γ which corresponds to a crossing of a canonical period path in F will then reproduce ϕ' with a multiplier $e^{c_i p_i}\mu_\phi$, p_i and μ_i being the corresponding moduli of periodicity of w_i and the multiplier of ϕ respectively. By a proper choice of the c_i it is evident that any desired multiplier system may be assigned to ϕ'.

In order that the form ϕ be expressible as a Poincaré series it is further necessary that it vanish in the parabolic fixed points. After a somewhat lengthy, but interesting analysis, a conclusion is reached similar to that in case $p = 0$, namely — every unimultiplicative form of dimension d satisfying the convergence condition, which vanishes in the parabolic points, and only such a form, is expressible as a Poincaré series; and every automorphic form with arbitrary poles, none of which occur at parabolic points, is expressible as a Poincaré series plus an integral automorphic form.

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LORIA'S SPECIAL PLANE CURVES.

Spezielle algebraische und transcendente ebene Curven, Theorie und Geschichte. Von Dr. GINO LORIA. Autorisierte, nach dem italienischen Manuscript bearbeitete deutsche Ausgabe von FRITZ SCHÜTTE. Leipzig, B. G. Teubner, 1902. 8vo., xxi + 744 pp. + 17 plates containing 174 figures.

ABOUT ten years ago the Royal Academy of Sciences at Madrid offered its triennial prize to be awarded upon the thirty-first of December, 1894, for "An ordered account of all curves of any kind which had received special names, and a further short account of their form, their equations, and their inventors." To this prodigious question no response seems to have come. Three years later the question was repeated. Professor Loria presented his researches, which were received

in a most flattering manner, as well they might be. These researches, in a somewhat altered form, carefully worked out and translated into German, form the contents of the large volume here under review.

Although the work is encyclopædic and all-inclusive so far as it goes, it does not cover the entire question set by the Academy at Madrid. Those curves, which are made up of arcs of other curves, and which therefore, though often of great use in applied mathematics or in the theory of the fine arts, are not representable by a single analytic equation, have purposely been omitted. No reference has been made to curves in space. The investigation of these must be left for a separate volume, which it is to be hoped Professor Loria will have time and inclination to write. In all other respects the author has covered the immense field of special curves whether named or not named. Thus, while in one way he has failed to compass the extent indicated in the announcement of the prize question, in another way he has exceeded the limits.

As the investigations on special curves often long antedate the *Jahrbuch* and lie hidden in doctors' theses, in essays, addresses, letters of a private correspondence, and in larger treatises where they are introduced only for some immediate special purpose, there can be little doubt — as the author himself modestly states — that some of the curves have not been attributed to their earliest inventors and that others may have been overlooked entirely. To find most of the special curves is not easy; to find them all is impossible. But probably no one person will detect more than a few corrections or additions to this work of Professor Loria, who has made such a profound and long-continued study of the field. Essential improvements must be the result of years and of many collaborators.

The method of treatment pursued by the author is, with the rarest exceptions, analytic. In the case of some curves, like the cissoid and conchoid, which date back to antiquity, a limited amount of geometric reasoning has been employed. For the most part, however, even these oldest curves have been handled analytically by the methods of rectangular or polar coördinates, or both. This method of treatment is to be commended, for only in this manner could be given anything approaching such a complete and systematic treatment as would interest modern readers. The large number of foot-notes, referring to the places where the original treatment of the curves may be

found, will be of invaluable service to those who delight in antiquarian methods or research. Most readers will find in the volume before us enough to satisfy them, except where the author has preferred to refer to some modern work in which a special curve is fully treated, rather than to give that treatment in detail. For example, the cardioid possesses numerous properties, of which only a few are given in the text, leaving the others to be studied in the elaborate thesis of Dr. R. C. Archibald.

Professor Loria's book will appeal in different degrees to a great variety of readers. That everything in the work should be interesting to a large number is inconceivable. Despite the careful and logical arrangement one can not fail to perceive that the problems are distinct and subject to distinct methods of treatment. The theory of special curves must necessarily be a collection of special theories, each applicable in general only to its particular curve. To this statement there are a few exceptions. If the curve happens to be unicursal, that is, expressible rationally in terms of a parameter, there are general theorems and methods which apply. Usually, however, one finds that he has at his disposal no other mathematical implements than elementary analytic geometry and the first principles of the differential and integral calculus. There must be some special motive or no one would read the book through as he might read a work on a more connected field of mathematics. On the other hand there is much here which will interest every one, especially one who teaches the elements of analytic geometry or of calculus. Every teacher knows the value of choosing problems which have interested mathematicians from early times down to the present day. Such problems are often easier for students to handle than the artificial ones invented purposely for exercises. The early mathematicians were not refined experts in analysis, having no great theories on which to draw. So they were forced to invent simple curves with simple properties immediately connected with simple geometric constructions, and these they treated by elementary means. The result is that this work done early in the development of mathematical science furnishes problems valuable in mathematical teaching.

This material has been collected by Professor Loria and put into a form easily accessible to all. For example the chapter on the cissoid of Diocles contains, in addition to the deduction of the equation of the curve, the following properties which,

taken collectively, form a series of interesting problems easy to prove by elementary methods. "The pedal of the parabola with respect to the vertex is a cissoid. The locus of points symmetric to the vertex of a parabola with respect to a (moving) tangent is a cissoid. If one parabola rolls externally upon an equal parabola the locus of the vertex is a cissoid. The polar reciprocal of a cissoid with respect to a circle about the cusp as center is a semi-cubical parabola. The envelope of the common chords of a cissoid and its (moving) circle of curvature is a similar cissoid of which the ordinates for corresponding abscissas are $\frac{1}{3}\cdot(\frac{1}{3})^{\frac{1}{2}}$ as great. The area between the cissoid and its asymptote is three times the area of the generating circle (Fermat; geometric interpretation by Huygens communicated to Wallis in a letter). The center of gravity of this area divides the distance between the cusp and the asymptote in the ratio 5 : 1. The volume generated by the revolution of the cissoid about its asymptote is equal to that of the ring generated by the same revolution of the generating circle (Sluse). The difference between the lengths of the cissoid and its asymptote when the ordinates y and $-y$ become infinite, approaches a finite quantity which may be evaluated by means of logarithms."

Here are eight theorems — four in analytic geometry, four in elementary calculus. These theorems are connected with such names as Diocles, Fermat, Huygens, Wallis, and Sluse; and the footnotes refer to places where these and further theorems may be found. Similar treatment is given to other curves including the cardiod, hypocycloid of three cusps, conchoid of Nicomedes, catenary, and tractrix. The student might well make the acquaintance of some of these famous curves early in his career. They are certainly more interesting than purely artificial loci. Then too there are curves in Professor Loria's book which are discussed by means of hyperbolic and elliptic functions and offer good exercise in their manipulation.

Most of the important theorems are printed in heavy faced type. They stand out on the text with an emphasis more than italic and may easily be found by rapidly turning the pages. A valuable and pleasing feature of the work is the elaborate plots of the important curves. There are one hundred and fifty numbered figures; but, as many of the numbers are subdivided, there are actually about one hundred and seventy-five different plots. The volume ends with an index of names, an index of subjects, and a special index for the plates. This careful atten-

tion to the form in which the volume is offered to the public is especially welcome in this case as it greatly facilitates rapid reference to any particular curve.

We now pass to a more detailed examination of the work. The first section of three chapters and thirteen pages, deals with the straight line, circle, and conic sections. Naturally the treatment is cursory, being in fact merely a few historical notes upon the most important ideas such as the geometry of a ruler, of a parallel ruler, or of the compasses. The author then passes immediately to higher plane curves.

The second section contains fourteen chapters reaching to the ninety-fourth page of the text and treating of cubics in point coördinates. The classification into five types (Chapter I) and the general treatment of unicursal (rational) cubics (Chapter II) are the most straightforward, neat and elementary that we know. The author, relying on the apology that his treatment is encyclopedic, gives F. W. Newman's classification of cubics into forty-two subtypes, each bearing a botanical or architectural name. It seems as if in this case a reference to the original memoir would be sufficient. In its place might have been given an account of that omitted but interesting work of Möbius which treats curves by means of their projection on a sphere. The circular curves, the cissoid with its generalization, the cubical parabola and folium of Descartes, the oblique and right strophoid, Sluse's conchoid, Rolle's curve, and the various witches, fill up the main part of the section. The two concluding, like the two beginning chapters, are the most interesting. They treat the cubic curves which serve for the trisection of an angle and the duplication of a cube, thus solving in a manner those two celebrated problems of antiquity, of which at least one is supposed to be of Pythian origin.

At this point we enter the only two serious complaints to be made against Professor Loria's work. First, there is practically no attention paid to line-coördinates, homogeneous or non-homogeneous. To be sure in earler times these coördinates were unknown. It is only within the last century or half-century that they have become of importance. Hence there must be very few special curves defined by means of them. Yet in any extended treatise, even on special curves, these coördinates merit more attention than the author gives them — as they also do in elementary instruction where, they can serve as the most simple and important examples of envelopes. We

see in Professor Loria's work numerous cases of isolated or acnodal points; but isolated tangents seem to be lacking. This geometric phenomenon is however more worthy of note as it seems more peculiar and more confusing than in the case of points. This first objection may be only half-hearted, but not so our second.

A more important objection lies in the fact that no mention is made in the text or indexes of Abel or his celebrated theorem which yields beautifully and powerfully some of the most interesting properties of higher plane curves. That the theorem applies in general is no hindrance to its use on special curves. The lack of it is not particularly felt when dealing with rational curves of the third and fourth orders. The author obtains the congruences, which would be given by Abel's theorem, by another means — that of the rational representation. But in any work on algebraic curves, special or otherwise, the total omission of this theorem, which is perhaps the greatest single theorem in the subject, seems scarcely conceivable.

The third section extends to page 218 and contains sixteen chapters. To the first three chapters, on classification of quartics, unicursal quartics, and elliptic and bicircular quartics, the remarks made upon the first two chapters of the preceding section apply. The absence of Abel's theorem begins to be felt. The succeeding chapters treat of Perseus's spiral, Nicomedes's conchoid, three-cusped curves of the fourth order and in particular the hypocycloid, cartesian ovals, polyzomals of the fourth order, rational curves with a node at which the tangents to the two branches are coincident, conchals, cassinian ovals, curves with three double points at each of which both branches possess inflections, the mussel-line, a quartic for trisecting angles, quartics which may be obtained in simple manners from conics. The mussel-line, although not especially interesting for its geometric properties, deserves a passing note because among a number of other things it was invented by Dürer (1525) for use in the fine arts and was said by him "to serve many a useful purpose."

The titles of the chapters of the first three sections have been quoted so fully merely to indicate the vast amount of detail, the great care and exhaustiveness with which the volume before us has been compiled. There appear a number of names doubtless quite unknown to most readers. In many cases the curves themselves possess no very great interest and

are passed by with but short notice. Some of the chapters are less than two pages in length; others extend over more than a dozen. In the following sections only the more important or better known curves will be mentioned.

Section IV deals with special curves of order greater than four. The orders reach nine and even twenty-five in the case of some curves connected with lemniscate functions. Probably the most interesting curve is Watt's, derived from Watt's parallelogram. The other names such as astroids, scarabeans, nephroids, atriphtaloids probably represent so little to the mind as to be not worth quoting.

To this point the curves have so been defined as to possess a definite order. The author next passes to curves which may have any desired order. Among these are the parabolas and hyperbolas of order n, the pearl curves, Cayley's general polyzomals defined by $\Sigma_i \sqrt{U_i} = 0$, ovals, curves for dividing an angle in a given ratio, self-polar curves, and curves obtained by inversion (anallagmatic). There are at the end of the section chapters, which possess a greater interest, upon algebraic curves whose rectification depends on assigned functions. The author gives applications to curves of which the length may be expressed in terms of arcs of an ellipse or lemniscate. Taken in its entirety this fifth section is one of the longest in the book; it contains seventeen chapters and over one hundred and fifty pages.

The sixth section is the longest, with its twenty-five chapters and almost two hundred pages, and the subject, special transcendental curves, is one of the most interesting. Among others are found curves for squaring the circle, spirals of various species, the whole trochoid family whether the base upon which the circle rolls be straight line or circle, the tractrix, catenary, cross-ratio or W-curves of Klein and Lie, lines of Mercator and Sumner, curves assumed by one-dimensional elastic bodies, the herpolhode, and some other curves of mathematical physics. Here, too, is a short account of the crinkly or otherwise amorphic curves. Strangely enough there seems to be no treatment of the Γ-curve in the text and the name does not occur in the index. The omission appears extraordinary as that curve not only possesses interesting properties of itself, but aids in the representation of many functions among which may be cited Euler's integrals of the second kind. The whole section is a veritable treasury of problems of a purely geometric nature

solvable by the use of the simpler transcendental functions, and as such it ought to appeal to those teachers of calculus who prefer not to draw very largely on physics for their problems. There are also excellent examples in the use of intrinsic coördinates and an appendix is devoted to showing how to pass from an equation in intrinsic to the corresponding equation in cartesian coördinates. The intrinsic equations are certainly interesting and important, and it is to be hoped that they will find greater recognition now that Professor Cesàro's work has appeared in German under the title Natürliche Geometrie.*

The seventh and concluding section, in twelve chapters and about one hundred and twenty pages, treats the very general question of curves which may be obtained by certain laws from a given curve. The generalized tractrix, evolutes, involutes, glissettes, parallel and radial curves, caustics, pedals and antipedals, and finally the derivative and integral curves derived by $y = f'(x)$ and $y = \int f(x)dx$ where $f(x)$ itself is defined by an equation $F(x, y) = 0$, form some of the topics. The last is worthy of especial mention owing to the fact that in general the derivative and integral curves are discussed only in the special case where the function $f(x)$ is a polynomial in x.

Although we have reached the end of the book proper, a note of a dozen pages under the unpromising title: Review of the historical development of the theory of plane curves, contains for advanced students a subject of more interest and stimulus than the rest of the volume taken together. This subject is the author's recent work on a class of curves which he calls *panalgebraic* and which are in general transcendental. The last seven pages of the note form a summary of Professor Loria's memoir, originally published in the *Prager Berichte*, and more recently reprinted with additions in the second volume of *Le Matematiche Pure ed Applicate*. Owing to the restricted American circulation of these periodicals the researches on panalgebraic curves are probably not well known among us, and we may be permitted in closing this review to go into detail upon the note with which the author closes his book.

Very truly he says: "What we need is a theory which will include most if not all of the known transcendental curves." Such is his theory of panalgebraic curves: for there are only about a half-dozen known curves which fail to fall under this

* See Dr. Snyder's review in the BULLETIN for April, 1903, pp. 349–357.

classification. The author considers a function $F(x, y, y')$ algebraic in each of the three variables x, y, y'. Without loss of generality we may assume that the function F is rational and integral and that the polynomials in x and y, which serve as coefficients of the powers of y', have no common factor $\phi(x, y)$. Write

$$F(x, y, y') \equiv \sum_{r=0}^{r=n} f_r(x, y) y'^{n-r} = 0. \qquad (1)$$

If the order of the polynomial F regarded as a function of x and y be ν, then ν is the order of that polynomial $f_r(x, y)$ which has the highest order.

Definition: A curve which satisfies a differential equation of the form (1) is said to be *panalgebraic*. The degree of the curve is said to be n, the degree of the polynomial F in y'. The rank of the curve is said to be ν, the degree of the polynomial F in x and y.

It is evident that the panalgebraic curves are to be regarded as the members of families, each of ∞^1 members: for the integral of equation (1) contains an arbitrary constant. An algebraic curve may be regarded as a panalgebraic curve of degree zero. The following theorems are fundamental in the theory.

I. If a curve is panalgebraic when regarded as a locus, it is panalgebraic when regarded as an envelope, and conversely.

II. In a family of panalgebraic curves of degree n and rank ν there are n curves which pass through each point of the plane and ν which touch each line.

III. In a family of panalgebraic curves (n, ν) there are $m\nu + n\mu$ which touch a given algebraic curve of order m and class μ.

IV. The points of tangency of the tangents drawn from a given point P to a panalgebraic curve (n, ν) lie on an algebraic curve of order $\nu + n$, of which the given point P is an n-fold point.

V. The tangents drawn at the points of intersection of a panalgebraic curve (n, ν) and a given line p touch an algebraic curve of class $n + \nu$, of which the given line p is an ν-fold tangent.

VI. The cusps of a panalgebraic curve lie on an algebraic curve.

VII. The inflection tangents of a panalgebraic curve touch an algebraic curve.

VIII. The system of all panalgebraic curves is invariant under the group of all contact transformations.

IX. The orthogonal trajectories of a family of panalgebraic curves form a family of panalgebriac curves of the same degree and rank.

X. There exist for panalgebraic curves certain derived algebraic curves analogous to the steinerian and hessian. Hence:

XI. The inflection points of a panalgebraic curve lie on an algebraic curve.

XII. The cuspidal tangents of a panalgebraic curve touch an algebraic curve.

When one considers that some relatively simple panalgebraic curves, such as the epicycloid in case the radii of the two circles are incommensurable, and the herpolhode practically fill completely an entire region of the plane, it is difficult not to be surprised at the simplicity of the results. For example the logarithmic spiral is of degree 1 and rank 1. The application of theorem IV yields the following result: The points of tangency of the tangents drawn from a point P to a logarithmic spiral lie upon a conic which passes through P. The application of the same theorem to Lissajous's curves ($n = 2$, $\nu = 2$) yields: The points of tangency of the tangents drawn from a point P to a Lissajous's curve lie upon a quartic which passes twice through the point P. A similar result obtains for the epicycloid ($n = 2$, $\nu = 4$) except that the locus is a sextic. Two general facts, first the dual property (theorem I), second the invariance under contact transformations (theorem VIII), will surely add to the simplicity and interest of any future detailed theory of panalgebraic curves.

In conclu ion we can not only express our thanks to Professor Loria for his volume on *special* curves and wish it success, but we can, with a pleasure the greater by contrast, recognize in him the founder of a theory of curves more *general* than any heretofore known. We can most heartily join in his closing hope that the present century will bring forth the much desired general theory and classification of transcendental plane curves. It is reasonable to assume that this work of his can not fail to be one of the most important advances toward the full realization of that hope.

EDWIN BIDWELL WILSON.

ÉCOLE NORMALE SUPÉRIEURE, PARIS,
December 30, 1902.

SHORTER NOTICES.

Théorie élémentaire des Séries. Par MAURICE GODEFROY, Bibliothécaire de la Faculté des Sciences de Marseille, avec une préface de L. SAUVAGE. Paris, Gauthier-Villars, 1903.

La Fonction Gamma. Par M. GODEFROY. Paris, Gauthier-Villars, 1902.

THE text-book literature of European mathematics has, since Euler's day, been blessed with numerous and most excellent books treating the elementary convergent processes from the algebraic standpoint. In such books we find a systematic application of the algorithm of inequalities to such questions as the convergence of infinite series, products and continued fractions and, after Cauchy's day, uniform convergence and termwise differentiation and integration of series.

These books which used to go by the generic title of books on algebraic analysis * afford an admirable introduction to a *theoretic* course in the calculus, in that they present its basal ideas in a form at once most concrete and comprehensible to the beginner.

It has been the custom of continental text-book writers on the calculus in recent years to incorporate such of this matter as relates to the more essential notions concerning infinite series in the initial chapters of the differential calculus. This course, while it has no doubt been dictated by necessity, seems from some points of view unfortunate.

It has thus come about that since Cauchy's Analyse algébrique few books of similar purpose have appeared in French and of these the only one that need here be mentioned, Tannery's Théorie des fonctions d'une variable, while admirable in every way and destined not soon to be supplanted, is perhaps too extensive for the beginner.

M. Godefroy has given us a book on the elementary theory of series that has much to recommend it. The exposition has the traditional French clearness and evinces a pedagogic insight that one would only expect in a writer accustomed to actual instruction.

* This name, apparently coined by Cauchy, is still used in Italy.

The book begins with a short (too short) account of irrational numbers defined by a "section" and establishes certain fundamental theorems due to Cauchy concerning the equality of the limits

$$\lim_{n=\infty}(x_{n+1}-x_n)=\lim_{n=\infty}(x_n/n),\quad \lim_{n=\infty}(x_{n+1}/x_n)=\lim_{n=\infty}\sqrt[n]{x_n},\ \text{etc.}$$

After a section on continuous functions of a single variable, series with positive terms are taken up and the powerful and elegant test of Kummer is established.

This remarkable criterion, which contains as special cases the geometric test, Raabe's test and the test of DeMorgan-Bertrand, admits of a very simple proof;* it necessitates, however, the construction of divergent scales of positive numbers. Our author, however, gives an independent deduction of the geometric test and after establishing the familiar theorems concerning conditionally convergent series and sundry theorems of Cauchy and Mertens concerning double series proceeds to the consideration of series whose terms involve a variable parameter.

The regional property of uniform convergence is defined and illustrated, unfortunately without any graphs, and Weierstrass's m-test is established. The rest of the book is devoted to power series and the more important functions defined by them.

The statement of Abel's theorem concerning the interval of convergence of a power series is open to the objection that our author fails to distinguish — although the distinction is made sharply enough in an earlier part of the book — between a suite of numbers that are always finite and one that has an upper and lower boundary.

The demonstration of the term by term differentiability of a power series, apparently due to Biehler† is very simple and has the merit that it goes back to first principles; this result might however have been more readily deduced from the continuity of such series.

Noteworthy features of the book are the proof of the existence theorem for homogeneous linear differential equations with analytic coefficients and the proof of the transcendence of the natural base e. The collection of excellent examples that fol-

* Cf. Cesàro: *Nouvelles Annales*, vol. 7, 3d series, p. 406, the proof in the text is however sufficiently simple.

† *Nouv. Annal.*, vol. 7, 3d series, p. 200.

lows each chapter will be found useful and the bibliography appended seems adequate.

The last chapter of the book is devoted to the gamma function and the auxiliary functions of Prym. It is condensed from La function gamma: Theorie, Histoire, Bibliographie of our author, and although the available analytic processes are necessarily restricted, the proofs are elegant and compact.

M. B. PORTER.

Höhere Analysis für Ingenieure. Von Dr. JOHN PERRY. Autorisierte deutsche Bearbeitung von Dr. ROBERT FRICKE und FRITZ SÜCHTING. Leipzig, B. G. Teubner, 1903. 8vo., viii + 423 pp.

IT is interesting to note in what numbers the last few years have produced treatises on the differential and integral calculus which have for their aim the fulfilment of the needs of some special class of students. For instance, there are for students of physics the elaborate treatise of Boussinesq and the smaller book by H. A. Lorentz, for chemists the work of Nernst and Schoenfliess recently translated into English by Young and Linebarger, for students of political economy a small primer by Irving Fisher, and for engineers the work of Perry. The question naturally arises whether so much subdivision in the study of calculus is necessary. It will prove a matter of serious inconvenience if each specialist must have a special course and a special book to suit his needs. There can be little doubt that the present tendency to this subdivision is due partly to the fact that mathematicians are apt to wish too selfishly to make their elementary courses strictly mathematical instead of practical, and in this respect we hope they will mend. A great part of the difficulty, however, is due to the inertness of the students of special branches, who wish to learn so much and only so much of calculus as appears to them necessary for their immediate needs. The short-sightedness of this attitude renders it dangerous. To-day many a chemist or economist, whose elementary education was finished a decade ago, complains of experiencing inconveniences because he did not study calculus. Perhaps before twenty-five years are past those who now are trying to learn as little of it as possible will be wishing that they had not tried to economize so much. In the earlier years of instruction time is not so precious as later, symbolic processes fix themselves more readily upon the mind which

later will be occupied more with thought. Thus a student who leaves his undergraduate course without a thorough knowledge of the purely formal manipulation of mathematics seldom can make good his loss.

It seems to us that one who studies Professor Perry's book will run just this risk of acquiring an insufficient formal knowledge of calculus. The author states that "the young engineer cannot be practiced too thoroughly in the mere simplification of algebraic and trigonometric relations, including those which contain complex quantities." The statement applies as well to the calculus and casts considerable doubt upon the efficacy of the author's arrangement of his book, in which he leaves such simple rules of operation as the differentiation of a sum, product, or quotient of two functions or of a function of a function, until the last chapter (p. 305) to be classed under the title, "More difficult exercises and theorems." To whom the work will be valuable is hard to say. So full is it of applications to mechanics, electricity and magnetism, thermodynamics, and elasticity as to be practically unintelligible to one who has not had considerable training as an engineer; and so little stress does it lay on the formal side of calculus that even the engineer, far from obtaining a working knowledge of the subject, would do well if he learned so much as "not to be afraid of calculus when he saw it."

We pass over the surprisingly few offenses against accuracy in demonstrations and definitions and turn our attention to a few features which might advantageously be copied in other elementary text-books on calculus. The first is the omission of indeterminate forms $0/0$, ∞/∞, $0\cdot\infty$, etc., of which the rigorous treatment is a matter of considerable difficulty. It is doubtful if students even understand what they are doing when they differentiate numerator and denominator until they find a determinate result. The whole subject may well be postponed until after treating the algebraic operations upon series. Secondly the introduction of the simpler differential equations and the methods of integrating them affords a pleasant and valuable change from the lengthy treatment of the formal integration of a rational function or a function rational in the domain $(x, \sqrt{ax^2+bx+c})$ with which most text-books insist on tormenting the students notwithstanding that a table of integrals like Professor B. O. Peirce's serves every purpose. Meanwhile, despite the fact that almost every application of cal-

culus involves a differential equation, the students remain unable to treat even the equation of harmonic motion and completely ignorant of the necessity and significance of "determining the arbitrary constants." Thirdly the author introduces Fourier's series long before he reaches the chapter on "more difficult exercises and theorems" in which he first states the rule for differentiating a product. Curious as this arrangement may seem, we wish to use it to call attention to the value and the possibility of early introducing the expansion of a function into a Fourier's series. Mathematicians as a rule are so preoccupied with contemplating the difficulties encountered in proving the expansibility of an arbitrary function into a Fourier's series that their students study calculus three hours a week for two full years, one hundred and eighty lessons, without hearing of any such series. Yet so valuable are these series in physics and astronomy that they might well be introduced, even crudely, as Professor Perry introduces them, early in the study of calculus. The actual determination of the coefficients in a case of the simpler functions is not difficult and furnishes a practical problem in integration.

Such are some of the advantages and disadvantages of Professor Perry's *Calculus for Engineers*. We can heartily recommend to any teacher the perusal of this book. In many places he will find his eyes opening to ways in which the treatment of difficulties, unsuspected and unappreciated by the student, may be replaced by questions of practical use and interest without endangering the career of any possible future pure mathematician. It will be only by a rational compromise on both sides that the inconvenience of the present tendency to different courses and different books for each class of specialists can be avoided without harm to any.

E. B. WILSON.

Leçons sur les Fonctions Méromorphes. Par ÉMILE BOREL. Recueillies et rédigées par LUDOVIC ZORETTI. Paris, Gauthier-Villars, 1903. 8vo., vi + 122 pp.

THE Leçons sur les Fonctions Méromorphes is a natural sequel to the author's Leçons sur les Fonctions Entières which appeared about three years ago. The results are in great part mere generalizations, to the case of functions possessing poles, of the results earlier obtained for functions possessing no singularities in the finite region of the complex plane. Considerable

attention is paid to numerical equalities and inequalities. The work draws its inspiration from Weierstrass, Du Bois-Reymond, and especially from Cauchy. It is distinctly of the modern French School, however, for with the exceptions of Mittag-Leffler and Lindelöf, who draw from the same source, the names that one meets are Picard, Poincaré, Hadamard, Borel and other less-known French mathematicians.

The chief topics treated are Mittag-Leffler's theorem that a meromorphic function is the sum of an integral (transcendental) function and a series of rational fractions; Weierstrass's theorem that a meromorphic function is the quotient of two integral functions; Hadamard's investigation on Taylor's series with applications to meromorphic functions and to the zeros of integral functions; the generalization of Picard's theorem to the statement that any meromorphic function must take on every value, except possibly two, an infinite number of times; and series of rational fractions, with ordinary and extraordinary distributions of poles, as applied to the study of meromorphic functions. Four notes, two of which Mr. Zoretti contributed, bring the subject so far up to date as to include some memoirs not published when the book went to press.

This little book, like the others, is of great value in putting us abreast of the present state of a particular branch of mathematics. So carefully has the matter been chosen and so clearly has Mr. Zoretti written that only a very limited knowledge of the theory of functions is presupposed. This pedagogical method is peculiarly French. It quickly places the student in absolute command of an advanced field so that he may commence his investigations at once without that vast auxiliary knowledge which some consider necessary. By such a work as these Leçons the whole mathematical world is given the same advantage.

E. B. WILSON.

NOTES.

THE April number (volume 25, number 2) of the *American Journal of Mathematics* contains the following papers: "The double-six configuration connected with the cubic surface, and a related group of Cremona transformations," by EDWARD KAS-

NER; "Untersuchungen über lineare Differentialgleichungen 4. Ordnung and die zugehörigen Gruppen," by SAUL EPSTEEN; "The logic of relations, logical substitution groups, and cardinal numbers," by A. N. WHITEHEAD; "On differential equations belonging to a ternary linearoid group," by F. E. ROSS; "On a certain group of isomorphisms," by J. W. YOUNG.

THE April number (second series, volume 4, number 3) of tne *Annals of Mathematics* contains: "The cardioid and tricuspid: quartics with three cusps," by R. C. ARCHIBALD; "Note on a partial differential equation of the first order," by E. D. ROE, JR.; "On a generalization of the set of associated minimum surfaces," by A. S. GALE; "Twisted quartic curves of the first species and certain covariant quartics," by H. S. WHITE; "On the characteristics of differential equations, Part I," by E. R. HEDRICK.

AT a meeting held in Boston, April 18, an association of teachers of mathematics in New England was organized for the improvement of methods of mathematical instruction in schools and colleges. Professor T. S. FISKE made an opening address on methods of improving the teaching of mathematics and Mr. W. T. CAMPBELL, of the Boston Latin School, read a paper on "Observational geometry." The following officers were elected: President, Mr. E. H. NICHOLS; vice-president, Professor W. F. OSGOOD; secretary and treasurer, Mr. F. P. DODGE, Roxbury Latin School, Boston, Mass. The association has at present about one hundred and sixty members. Copies of the constitution and of the circular letter issued as a call for the recent meeting can be obtained from the secretary.

AT the meeting of the London mathematical society on March 12, the following papers were read: "On the convergence of certain multiple series," by G. H. HARDY; "On certain sequences for determining the nth root of a rational number," by Mr. S. M. JACOB; "Note on the approximate calculation of the frequencies of a vibrating circular plate," by Professor H. LAMB; "On surfaces which have assigned families of curves as their lines of curvature," by Professor A. R. FORSYTH; "Note on a point in Hilbert's 'Grundlagen der Geometrie,'" by Mr. E. T. DIXON; "Extension of two theorems on covariants," by Mr. J. H. GRACE; "Note on double limits and on the inversion of a repeated infinite inte-

gral," by Professor T. J. I'A. BROMWICH; "On the representation of a group of finite order as an irreducible group of linear substitutions, and the direct establishment of the relations between the group characteristics," by Professor W. BURNSIDE.

THE section of the history of sciences of the international congress of historic sciences, which met in Rome in April, 1903, has decided to form an international commission on the organization of a congress of the history of sciences. The following officers have been chosen: President, J. TANNERY, Paris; secretaries, P. GIACOT, Turin, and G. LORIA, Genoa. Professor D. E. SMITH has been appointed the American representative. The next meeting of the congress will be held in Berlin, in September, 1906.

THE German association for the promotion of the teaching of mathematics and natural sciences held its twelfth general meeting at Breslau, June 1–5.

THE several American universities below offer during the academic year 1903–1904 advanced courses in mathematics as follows:

UNIVERSITY OF CALIFORNIA.—By Professor I. STRINGHAM: Logic of mathematics, two hours; Analytic non-euclidean geometry of space, two hours.—By Professor G. C. EDWARDS: Ordinary differential equations, three hours.—By Professor M. W. HASKELL: Theory of functions of a complex variable, three hours.—By Mr. A. W. WHITNEY: Theory of probabilities, two hours.—By Dr. D. N. LEHMER; Synthetic projective geometry, three hours.—By Dr. E. M. BLAKE: Line geometry, three hours.—By Dr. T. M. PUTNAM: Theory of numbers, three hours; Mathematical seminar, foundations of dynamics, two hours.

THE UNIVERSITY OF CHICAGO.—The following advanced mathematical courses, four hours weekly, are offered during the four quarters (*su*, *a*, *w*, *sp*) of the year beginning June 17, 1903:—By Professor E. H. MOORE: Theory of functions of real variables (*su*); Seminar (*a*, *w*, *sp*).—By Professor O. BOLZA: Theory of equations (*a*, *w*); Quaternions (*sp*); Elliptic functions, the Weierstrass theory (*a*); Applications of elliptic functions (*su*); Abelian functions (*w*, *sp*); Invariants (*su*).—

By Professor H. MASCHKE: Solid analytics (*sp*); Twisted curves and surfaces (*w*, *sp*); Invariants (*w*).—By Professor H. E. SLAUGHT: Advanced integral calculus (*aw*).—By Professor J. W. H. YOUNG: Solid analytics and determinants (*su*); conferences on the pedagogy of mathematics (*su*).—By Professor L. E. DICKSON: Theory of functions of a complex variable (*su*); Theory of functions (*a*, *w*).—By Mr. A. C. LUNN: The differential equations of mathematical physics (*su*, *sp*); Graphic methods in the teaching of elementary mathematics (*su*).—By Dr. S. EPSTEEN: History of elementary mathematics (*su*, 2 hours).—By Professor K. LAVES: Analytic mechanics (*a*, *w*). —By Professor F. R. MOULTON: Introduction to celestial mechanics (*w*, *sp*); Selected chapters of celestial mechanics (*su*).

CORNELL UNIVERSITY.—By Professor L. A. WAIT: Advanced analytic geometry (plane and solid), three hours; Advanced differential calculus, three hours.—By Professor G. W. JONES: Higher algebra and trigonometry, three hours.—By Professor J. MCMAHON: Higher plane curves, two hours; Potential function, two hours; Quaternions, two hours; Theoretical mechanics, two hours; Mathematical theory of sound. —By Professor J. H. TANNER: Theory of equations, two hours; Invariants, two hours.—By Dr. J. I. HUTCHINSON: Theory of functions, two hours; Elliptic and abelian functions, two hours; Projective geometry, three hours.—By Dr. V. SNYDER: Advanced integral calculus, two hours; Line geometry, three hours.—By Dr. W. B. FITE: Differential equations, two hours; Theory of groups, three hours.

HARVARD UNIVERSITY.—By Professor J. M. PEIRCE: Quaternions; Theory and application of tetrahedral coördinates; † The algebra of logic; † Finite differences; † The calculus of probabilities; † Linear associative algebra.—By Professors W. E. BYERLY and B. O. PEIRCE: Trigonometric series, spherical harmonics, and the potential function. By Professor W. E. BYERLY: Calculus (second course); Introduction to modern geometry and modern algebra; † Dynamics of a rigid body.—By Professor W. F. OSGOOD: The theory of functions (first course).—By Professor M. BÔCHER: † Infinite series and products; Algebra; † Linear differential equations. —By Dr. C. L. BOUTON: † The theory of numbers; † The elementary theory of differential equations; † Geometric transformations.—By Mr. J. K. WHITTEMORE: Celestial mechanics; † Hydromechanics.

These courses will involve three lectures a week throughout the year, except those preceded by †, which involve about half this number of lectures. Professor Bôcher, Dr. Bouton, and Mr. Whittemore also offer courses in reading and research in Differential equations, Continuous groups, and Differential geometry respectively.

STANFORD UNIVERSITY.—Professor R. E. ALLARDICE: Invariants, two hours, first semester; Definite integrals, two hours, second semester; Geometry of three dimensions, two hours; Theory of functions, three hours.—By Professor R. L. GREEN: Advanced coördinate geometry, two hours; Theory of equations, three hours.—By Professor L. M. HOSKINS: Theory of attraction and the potential function, two hours.—By Professor G. A. MILLER: Elementary theory of groups, three hours; Theory of numbers, two hours, second semester; History of mathematics, two hours, first semester; Seminar in the theory of groups, two hours.—By Professor H. F. BLICHFELDT: Determinants, two hours, first semester; Non-euclidean geometry, two hours, second semester; Differential equations, three hours.

NORTHWESTERN UNIVERSITY:—*First semester.*—By Professor T. F. HOLGATE: Linear systems of conics, two hours.—By Professor H. S. WHITE: Theory of functions, two hours; Plane cubics and quartics, three hours. *Second semester.*—By Professor T. F. HOLGATE: Theory of numbers, two hours.—By Professor H. S. WHITE: Elliptic functions, three hours; Projective geometry of surface and twisted curves, two hours.

YALE UNIVERSITY.—By Professor JAMES PIERPONT: Advanced calculus, three hours; Projective geometry, three hours; Theory of functions of real variables, three hours.—By Professor P. F. SMITH: Continuous groups, three hours; Higher analysis for engineers, two hours, first semester.—By Professor H. A. BUMSTEAD: Problems in mathematical physics, two hours.—By Dr. A. S. GALE: Differential equations and function theory, three hours.—By Dr. H. E. HAWKES: Higher algebra, two hours; Elliptic functions, three hours.—By Dr. W. A. GRANVILLE: Differential geometry, two hours.—By Dr. E. R. HEDRICK: Calculus of variations, three hours.—By Dr. E. B. WILSON: Analytic mechanics, two hours; Introduction to mathematical physics, two hours; Non-euclidean geometry, two hours.

THE several foreign universities and technical schools below offer during the present summer semester courses in mathematics as follows:

UNIVERSITY OF BERLIN. — By Professor J. KNOBLAUCH: Analytic geometry, four hours; Theory of partial differential equations, four hours; Theory of ray systems, one hour. — By Professor R. LEHMANN-FILHÉS: Differential equations, four hours; Exercises, one hour. — By Professor H. A. SCHWARZ: Integral calculus, four hours; Exercises; Applications of elliptic functions, two hours; Calculus of variations, four hours; Colloquium; Seminar. — By Professor F. SCHUR: Introduction to the theory of ordinary differential equations, two hours. — By Dr. E. LANDAU: Theory of numbers, four hours; Introduction to the theory of functions, four hours; Theory of Riemann zeta-functions, one hour. — By Professor G. FROBENIUS: Theory of algebraic equations, four hours; Seminar. — By Professor F. H. SCHOTTKY: Theory of abelian functions, four hours; Seminar. — By Professor G. HETTNER: Theory of potential, two hours.

UNIVERSITY OF CZERNOWITZ. — By Professor O. TUMLIRZ: Mathematico-physical seminar, two hours; proseminar, two hours.

UNIVERSITY OF GRAZ. — By Professor J. FRISCHAUF: Differential and integral calculus, and its application to geometry, five hours. — By Professor V. D. R. VON KOLLESBERG: Analytic and projective geometry of the plane, five hours; Mathematical seminar, two hours. — By Dr. J. STREISSLER: Descriptive geometry, three hours.

UNIVERSITY OF INNSBRUCK. — By Professor O. STOLZ: Theory of functions of a complex variable according to Cauchy and Weierstrass, with exercises in the mathematical seminar, three hours; Arithmetic, the theory of real numbers, with exercises in the mathematical seminar, four hours. — By Professor W. WIRTINGER: Higher algebra, three hours; Abelian functions, two hours; Seminar, two hours. — By Professor K. ZINDLER: Plane and solid analytic geometry, two hours; Line geometry, three hours; Seminar, one hour.

UNIVERSITY OF KÖNIGSBERG. — By Professor F. MEYER: Analytic geometry, four hours; Exercises in analytic geometry,

one hour; Introduction to higher geometry, four hours; Exercises in higher geometry, one hour.—By Professor A. SCHOENFLIES: Theory of differential equations four hours; Exercises in the mathematical seminar, two hours.—By Professor L. SAALSCHÜTZ: Determinants, two hours; Gaussian and other interesting series, four hours.—By Dr. F. COHN: Applications of the potential theory, three hours.—By Dr. T. VAHLEN: Differential calculus, four hours; Exercises in differential calculus, one hour.

UNIVERSITY OF MÜNSTER.—By Professor W. KILLING: Differential and integral calculus, I, three hours; Synthetic geometry, four hours; Selected topics from elementary mathematics, two hours; Exercises in the proseminar, two hours; Exercises in differential and integral calculus, one hour; Supplement to analytic geometry, one hour.—By Professor R. VON LILIENTHAL: Analytic geometry, I, four hours; Theory of determinants, with applications, four hours; Exercises in the mathematical seminar, one hour.—By Dr. M. DEHN: Theory of numbers, four hours; Higher mathematics for naturalists, three hours; Exercises in the theory and application of elliptic functions, one hour.—By Dr. J. KARST: Introduction to descriptive geometry.

UNIVERSITY OF PRAG.—By Professor G. PICK: Differential equations, three hours; Differential and integral calculus, two hours; Seminar, two hours.—By Professor J. A. GMEINER: Analytic geometry, three hours; Number congruences, two hours.—By Dr. W. WEISS: Elements of descriptive geometry, two hours.

UNIVERSITY OF VIENNA.—By Professor G. VON ESCHERICH: Elements of the differential and integral calculus, five hours; Exercises, two hours; Proseminar, one hour; Seminar, two hours.—By Professor F. MERTENS: Theory of numbers (continuation), five hours; Exercises in the mathematical seminar, two hour; in the proseminar, one hour; Theory of probabilities, three hours; Mathematical statistics, three hours.—Professor G. KOHN: Synthetic geometry (continuation), four hours, with exercises, one hour; Theory of invariants, with geometric applications (continuation), two hours.—By Dr. E. BLASCHKE: Introduction to mathematical statistics, II, three hours.—By Dr. DAUBLEBSKY VON STERNECK: Algebra, three

hours.—By Dr. K. CARDA: Selected topics from the theory of contact transformations, two hours.—By Dr. J. PLEMELJ: Theory of potential, with applications (continuation), two hours.

UNIVERSITY OF WÜRZBURG.—By Professor F. PRYM: Integral calculus, six hours; Exercises on integral calculus, in the proseminar, two hours; Selected chapters in the theory of functions, in the seminar, two hours.—By Dr. G. ROST: Descriptive geometry, II, four hours; Analytic and synthetic geometry of conics, four hours; Application of the infinitesimal analysis to geometry, four hours; Selected chapters from elementary mathematics in the proseminar, two hours.

UNIVERSITY OF BASEL.—By Professor H. KINKELIN: Algebraic analysis, three hours; Geometric applications of differential calculus, three hours; Definite integrals, two hours; Theory of probabilities, three hours.—By Dr. R. FLATT: Line geometry, two hours.

UNIVERSITY OF BERN.—By Professor J. H. GRAF: Bessel's functions, three hours; Elliptic functions, three hours; Function theory, two hours; Differential equations, two hours; Differential and integral calculus, two hours; Mathematics of finance and insurance, two hours; Seminar, with Professor Huber, two hours; Seminar (insurance) with Professor Moser, one hour.—By Professor G. SIDLER: On elliptic arcs, whose difference is rectifiable, two hours.—By Professor G. HUBER: Introduction to the theory of algebraic surfaces, two hours.—By Professor E. OTT: Differential calculus, two hours; Analytic geometry, I, two hours.—By Dr. A. BENTELI: Elements of descriptive geometry, four hours.—By Dr. L. CRELIER: Synthetic geometry of space, two hours.

UNIVERSITY OF GENEVA.—By Professor C. CAILLER: Differential and integral calculus (continuation), three hours: Seminar, two hours.—By Professor H. FEHR: Descriptive and projective geometry, two hours; Algebra; General theory of equations, two hours; Exercises in the calculus (with Professor Cailler), two hours, and in algebra and geometry, two hours.—By Dr. D. MIRIMANOFF: Dirichlet's problem, two hours.

UNIVERSITY OF LAUSANNE. — By Professor H. AMSTEIN: Differential and integral calculus, five hours; Exercises, two hours; Theory of elliptic functions, three hours; Calculus for naturalists, three hours; Selected topics from the integral calculus (definite integrals and series), two hours. — By Professor H. JOLY: Analytic geometry (continuation), two hours; Descriptive geometry (continuation), two hours; Outline of descriptive geometry, four hours; Theory of numbers, two hours.

UNIVERSITY OF ZURICH. — By Professor H. BURKHARDT: Linear differential equations, four hours; Vector analysis, two hours; Seminar, two hours; Mathematical treatment of periodic recurrences in nature, two hours. — By Professor A. WEILER: Analytic geometry, II, two hours; Descriptive geometry, three hours; Mathematical geography, two hours. — Map projections, two hours. — By Dr. F. KRAFT: General theory of assemblages, four hours. — By Dr. E. GUBLER: Theory of numbers, two hours; Political arithmetic, with exercises, two hours.

CARLSRUHE TECHNICAL HIGH SCHOOL. — By Professor R. HAUSSNER: Elementary and analytic geometry of the plane and space, three hours, with exercises, one hour; Synthetic geometry, two hours, with exercises, one hour. — By Professor A. KRAZER: Higher mathematics, I, six hours, with exercises, two hours. — By Professor F. SCHUR: Descriptive geometry, four hours, with exercises, four hours. — By Professor L. WEDEKIND: Higher mathematics, II, two hours. — By Dr. G. HAMEL: Elements of higher mathematics, four hours, with exercises, one hour. — By Dr. W. LUDWIG: Theory of projection, two hours, with exercises, two hours.

MUNICH TECHNICAL HIGH SCHOOL. — By Professor A. VON BRAUNMÜHL: Algebraic analysis, with exercises (continuation); projective geometry synthetically treated, with exercises (continuation); mathematico-historical seminar. — By Professor S. FINSTERWALDER: Higher mathematics, II, with exercises; non-euclidean geometry. — By Professor W. VON DYCK: Partial differential equations of mathematical physics; colloquium, with Professor Finsterwalder. — By Professor L. BURMESTER: Descriptive geometry, with exercises (continuation). — By Professor S. GÜNTHER: Theory of potential with its application to geophysics.

STUTTGART TECHNICAL HIGH SCHOOL. — By Professor W. BRETSCHNEIDER: Elementary mathematics. — By Professor K. MEHMKE: Descriptive geometry; seminar — By Professor K. REUSCHLE: Differential and integral calculus; Plane analytic geometry; Seminar. — By Dr. E. WÖLFFING: Theory of curves; Partial differential equations.

THE current numbers of the *Revue générale des sciences*, beginning with volume 14, number 2, contain a very interesting series of articles in which, under the caption "The evolution of mechanics," P. DUHEM discusses the relations of theoretical mechanics to physical theories. The titles of the successive papers will give some idea of the wide field covered: I. Les diverses sortes d'explications mécaniques; II. La mécanique analytique; III. Les théories mécanique de la chaleur et de l'electricité; IV. Le retour à l'atomisme et au cartésianisme; V. Les fondements de la thermodynamique; VI. La statique générale et la dynamique générale; VII. Les branches aberrantes de la thermodynamique.

PROFESSOR P. GORDAN, of Erlangen, has been made an honorary member of the University of Juriev (Dorpat). Dr. I. IVANOF and Dr. W. STANIEVICH have been made professors of mathematics at the St. Petersburg Polytechnic School. Dr. L. NATANSON has been promoted to a full professorship of mathematics at the University of Cracow.

PREBENDARY W. A. WHITWORTH, formerly fellow of St. John's College, Cambridge, has been appointed Hulsean lecturer for 1903.

MR. HERBERT KNAPMAN, first Smith's prizeman this year, has been appointed assistant lecturer in mathematics at University College, Reading, England.

MR. W. H. WAGSTAFF delivered the Gresham lectures on geometry this year, April 28–May 1.

PROFESSOR G. DARBOUX has recently been elected member of the bureau of longitudes.

PROFESSOR W. F. OSGOOD, of Harvard University, has been promoted to a full professorship of mathematics.

DR. ARNOLD EMCH, of the University of Colorado, has been promoted to a full professorship of graphics and mathematics.

DR. C. A. NOBLE has been promoted to an assistant professorship of mathematics at the University of California.

AT Lehigh University Professor ALEXANDER MACFARLANE delivered, April 20–30, a course of six lectures on the British mathematicians Kirkman, Babbage, Whewell, Dodgson, Stokes, and Rayleigh.

MR. F. G. WREN has been advanced from an assistant professorship to a professorship of mathematics at Tufts College.

PROFESSOR J. W. MASON, of the College of the City of New York, will retire from service at the end of the present year.

DR. N. M. FERRERS, for a long time one of the editors of the *Quarterly Journal of Mathematics*, died on January 31, 1903.

PROFESSOR JOSIAH WILLARD GIBBS, of Yale University, died at New Haven, April 28, 1903, of heart disease. He was born in New Haven, February 11, 1839, and graduated at Yale in 1858. After receiving the doctor's degree in 1863 from the same institution, he studied in Paris, Berlin, and Heidelberg. In 1871 he was appointed to the professorship of mathematical physics at Yale which he held to the time of his death. Professor Gibbs was a member of the Royal Society of London, of the National Academy of Sciences, and of many other learned bodies. Quite recently he joined the AMERICAN MATHEMATICAL SOCIETY. He was an authority of the first rank in thermodynamics and in the application of vector analysis to physical problems.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

AMODEO (F.). Appunti e riposte. Lettera aperta ad un geometra italiano. Napoli, 1902. 8vo. 10 pp.

——. Dai fratelli di Martino a Vito Caravelli. Napoli, 1902. 8vo. 64 pp.

BIBLIOGRAPHIE der deutschen naturwissenschaftlichen Litteratur. Herausgegeben im Auftrage des Reichsamtes des Innern vom deutschen Bureau der internationalen Bibliographie in Berlin. Vol. 3: 1903–4. Abteilung 1: Mathematik, Mechanik, Physik, Chemie, Astronomie, Meteorologie. No. 1. Jena, Fischer, 1903. 8vo. 48 pp. M. 9.00

BRADSHAW (J. W.). The logarithm as a direct function. 4to. (*Annals of Mathematics* (2) 4, pp. 51–62.)

BRAUNMÜHL (A. VON). Vorlesungen über Geschichte der Trigonometrie. Teil 2: Von der Erfindung der Logarithmen bis auf die Gegenwart. Leipzig, Teubner, 1903. 8vo. 11 + 264 pp. M. 10.00

BUCHERER (A. H.). Elemente der Vektoranalysis; mit Beispielen aus der theoretischen Physik. Leipzig, Teubner, 1903. 8vo. 6 + 91 pp. M. 2.40

COUTURAT (L.). See LEIBNIZ.

DINGELDEY (F.). See ENCYKLOPÄDIE.

ENCYKLOPÄDIE der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. III: Geometrie, herausgegeben von W. F. Meyer. Teil 2, Heft 1: F. Dingeldey, Kegelschnitte und Kegelschnittsysteme. Leipzig, Teubner, 1903. 8vo. Pp. 1–160.

ESTANAVE (E.). Essai sur la sommation de quelques séries trigonométriques. Paris, Hermann, 1903. 8vo. 112 pp. Fr. 6.00

GULDBERG (A.). See LIE (S.).

INTERNATIONAL CATALOGUE of scientific literature. First annual issue [for 1901]. A. Mathematics. Published for the International Council by the Royal Society of London. London, Harrison. Vol. VII: 1902 (November). 8vo. 16 + 201 pp.

JAHRAUS (K.). Das Verhalten der Potenzreihen auf dem Konvergenzkreise, historisch-kritisch dargestellt.. (Progr.) Ludwigshafen, 1902. 8vo. 56 pp.

JOUFFRET (E.). Traité élémentaire de géométrie à quatre dimensions et introduction à la géométrie à n dimensions. Paris, Gauthier-Villars, 1903. 8vo. 30 + 216 pp. Fr. 7.50

KASNER (E.). The cogredient and digredient theories of multiple binary forms. 4to. (*Transactions of the American Mathematical Society* 4, pp. 86–102.)

——. The generalized Beltrami problem concerning geodesic representation. 4to. (*Transactions of the American Mathematical Society* 4, pp. 149–152.)

LEIBNIZ. Opuscules et fragments inédits. Extraits des manuscrits de la bibliothèque royale de Hanovre par L. Couturat. Paris, Alcan, 1903. 8vo. 689 pp. (Collection historique des grands philosophes.)

LIE (S.). Ueber Integralinvarianten und Differentialgleichungen. (Ausgegeben von A. Guldberg und K. Störmer im Auftrage der k. Gesellschaft der Wissenschaften zu Kristiania und auf Kosten des Fridjof Nansen Fond.) Kristiania, 1902. 8vo. 73 pp. K. 2.00

LINDOW (M.). Die Nullstellen des allgemeinen Integrals der Differentialgleichung für die zugeordneten Kugelfunktionen. (Diss.) Halle, 1902. 8vo. 65 pp.

LINNEBORN (J.). Die Fokaleigenschaften der Gebilde zweiter Ordnung in der Riemannschen Raumform. (Diss.) Münster, 1902. 8vo. 26 pp.

LIPPITSCH (K.). Die Unverträglichkeits-Relation des Satzes vom goldenen Schnitte mit dem Gesetze der rationalen Indices, nachgewiesen am Rautendreissigflächner und regelmässigen Pentagondodekaeder. (Progr.) Leoben, 1902. 8vo. 12 pp., 1 plate.

MEYER (W. F.). See ENCYKLOPÄDIE.

MÜLLER (J. O.). Ueber die Minimaleigenschaft der Kugel. (Diss.) Göttingen, 1903. 8vo. 51 pp.

PAMPUCH (A.). Das Malfatti-Steiner'sche Problem. (Progr.) Strassburg i. E., 1902. 4to. 73 pp., 10 plates.

PASCAL (B.). Oeuvres complètes. Vol. 3: Traités divers de physique et de mathématiques. Table analytique. Paris, Hachette, 1903. 16mo. 502 pp. (Les principaux écrivains français.). Fr. 1.25

POINCARÉ (H.). La science et l'hypothèse. Paris, Flammarion, 1903. 12mo. 284 pp. Fr. 3.50

ROCQUIGNY-ADANSON (G. DE). Théorèmes sur les progressions arithmétiques; notes d'arithmologie; propositions de la théorie des nombres. Moulins, Auclaire, 1903. 8vo. 20 pp.

ROTHE. Lösung einiger Aufgaben über Flächenberechnungen mit Hilfe elliptischer Integrale. (Progr.) Nordhausen, 1902. 4to. 22 pp., 4 plates.

SEYBOLD (C.). Die Drusenschrift: Kitāb Alnoqat Waldawāir, "Das Buch der Punkte und Kreise." Nach dem Münchener und Tübinger Codex. (Festschrift.) Tübingen, 1902. 4to. 111 pp.

SEYLER (G.). Ueber die Erhaltung der Krümmungslinien bei Orthogonalprojektion. (Progr.) Passau, 1902. 8vo. 24 pp.

SPENCER (F.). Ueber Konchoiden. (Progr.) Schwerin, 1902. 4to. 11 pp.

STÖRMER (A.). See LIE (S.).

TROPFKE (J.). Geschichte der Elementar-Mathematik in systematischer Darstellung. Vol. I: Rechnen und Algebra. Leipzig, 1902. 8vo. 8 + 332 pp. M. 8.00

ZAHLER (R.). Das Abelsche Theorem für Grundkurven, die in Gerade und Kegelschnitte zerfallen. (Diss.) München, 1902.

II. ELEMENTARY MATHEMATICS.

AVERY (J. A.). Plane geometry by the suggestive method. Boston, Sanborn, [1903]. 12mo. 6 + 122 pp. Boards. $0.40

BACKHAUS (K.). See WIESE (B.).

BAKER (W. M.) and BOURNE (A. A.). Elementary geometry, complete. London, Bell, 1903. 12mo. 506 pp. Cloth. 4s. 6d.

BÉNÉZECH (L.). Cours d'algèbre. Livre I: Nombres algébriques (définition; opérations; applications). Rédigé conformément aux programmes officiels du 31 mai 1902, à l'usage des classes de troisième A et de quatrième B, des écoles primaires supérieures et professionnelles, et des candidats aux écoles nationales d'arts et métiers et aux écoles des mécaniciens des équipages de la flotte. Paris, Alcan, 1903. 18mo. 53 pp. Fr. 1.20

BIEL (B.). Mathematische Aufgaben für die höheren Lehranstalten, unter möglichster Berücksichtigung der Anwendungen, wie überhaupt der Verknüpfung der Mathematik mit anderen Gebieten zusammengestellt. Ausgabe für Realanstalten. Teil I: Die Unterstufe. Leipzig, Freytag, 1903. 8vo. 6 + 206 pp. Cloth. M. 2.50

BOURLET (C.). Cours abrégé d'arithmétique, contenant 1421 énoncés d'exercices. 3e édition, revue. Paris, Hachette, [1903]. 16mo. 414 pp. (Cours complet de mathématiques.) Fr. 2.50

BOURNE (A. A.). See BAKER (W. M.).

CRAWLEY (E. S.). Tables of logarithms, five-place; seven tables, with explanations. [New edition.] Philadelphia, Crawley, [1903]. 8vo. 32 + 76 pp. Cloth. $0.75

——. The elements of trigonometry and five-place tables, bound together. [New edition.] Philadelphia, Crawley, [1903]. 8vo. Cloth. $1.25

DAVID (M. S.). Beginner's algebra. London, Black, 1903. 12mo. 2s. 6d.

DEAKIN (R.). Euclid, books 5, 6, 11. London, Clive, 1903. 12mo. 152 pp. Cloth. (University tutorial series.) 1s. 6d.

DICKSON (L. E.). Introduction to the theory of algebraic equations. New York, Wiley, 1903. 12mo. 5 + 104 pp. Cloth. $1.25

DUPRET (H.). Leçons d'algèbre, à l'usage des élèves de la quatrième B et de la troisième A de l'enseignement secondaire des lycées et collèges. Arras, Segaud, [1903]. 4to. 48 pp.

FLETCHER (W. C.). See LACHLAN (R.).

FREYCINET (C. DE). De l'expérience, en géométrie. Paris, Gauthier-Villars, 1903. 8vo. 19 + 178 pp. Fr. 4.00

GIRNDT (M.). Raumlehre für Baugewerkschulen und verwandte gewerbliche Lehranstalten. Teil I: Lehre von den ebenen Figuren. Mit 207 der Baupraxis entlehnten Aufgaben. 2te Auflage. Leipzig, Teubner, 1903. 8vo. Cloth. M. 2.40

GODFREY (C.). See HOCEVAR (F.).

HOCEVAR (F.). Solid geometry; translated and adapted by C. Godfrey and E. A. Price. London and New York, Macmillan, 1903. 12mo. 6 + 80 pp. Cloth. $0.50

JIMÉNEZ RUEDA (C.). Programa de complementos de algebra y geometría. Madrid, Suárez, 1903. 8vo. 16 pp. (Universidad central.) Fr. 1.00

KRAUS (K.). Grundriss der Geometrie und des geometrischen Zeichnens für Lehrerbildungsanstalten. Mit einem Situationsplan. Wien, Pichler, 1902. 8vo. 2 + 238 pp. Cloth. M. 2.70

LACHLAN (R.) and FLETCHER (W. C.). Elements of geometry. London, Arnold, 1903. 12mo. 220 pp. Cloth. 2s. 6d.

LICHTBLAU (W.). See WIESE (B.).

LONDON matriculation model answers: mathematics. Papers in mathematics from January 1897 to September 1902. London, Clive, 1903. 12mo. 174 pp. (University tutorial series.) 2s.

LUDWIG (F. K.). Das logarithmische Rechnen. Leichtfassliche Darlegungen über das Wesen, die Berechnung und Anwendung der Logarithmen nebst zahlreichen Beispielen und ausführlichen Lösungen. Zum Selbstunterricht bearbeitet. Reichenberg, Sollors, [1902]. 8vo. 3 + 52 pp. M. 1.00

NEUFFER. Elementare ebene Oerter. (Progr.) Ulm, 1902. 4to. 64 pp.

PETCH (T.). Plane geometry adapted to heuristic method of teaching. London, Arnold, 1903. 12mo. 120 pp. Cloth. 1s. 6d.

PRICE (E. A.). See HOCEVAR (F.).

SCHULTZ (E.). Kurzgefasstes Lehrbuch der Körperberechnung für gewerbliche Schulen. Essen, Baedeker, 1903. 8vo. 4 + 50 pp. Boards. M. 1.00

——. Nebst Aufgabensammlung. Essen, Baedeker, 1903. 8vo. 4 + 99 pp. Boards. M. 1.70

SIMÓN Y MAYORGA (J.). Programa de nociones de aritmetica y geometría. Salamanca, Calón, 1902. 8vo. 18 pp.

WENTWORTH (G. A.). Logarithms, metric measures, and special subjects in advanced algebra. Boston, Ginn, 1903. 12mo. 141 pp. $0.25

WIESE (B.), LICHTBLAU (W.), und BACKHAUS (K.). Raumlehre für Lehrerbildungsanstalten. (In 2 Teilen.) Teil 2: Stereometrie und Trigonometrie (Körperlehre und Dreiecksrechnung). 4te Auflage, umgearbeitet und erweitert nach dem preussischen Lehrplan für die Lehrerseminare vom 1. VIII. 1901 und in der neuen deutschen Rechtschreibung gedruckt. Breslau, Hirt, 1903. 8vo. 207 pp. M. 2.25

III. APPLIED MATHEMATICS.

ALIX (L.). Curso de geometría descriptiva, accompañado de un atlas dividido en tres partes. Valencia, Alufre, 1902. 4to. 112 + 199 + 75 pp., 31 + 48 + 13 plates. Fr. 20.00

ARRHENIUS (S. A.). Lehrbuch der kosmischen Physik. In 2 Teilen. Leipzig, Hirzel, 1903. 8vo. 8 + 8 + 1026 pp., 3 plates. M. 38.00

ARSONVAL (D'). La rotation de la terre démontrée par le pendule de Foucault. Appareil des écoles, approuvé par Flammarion et Berget, présenté à l'Académie des sciences, le 17 novembre 1902. Les Châtelles, Raon-l'Etape, [1903]. 8vo. 32 pp., portrait.

BACH (C.). Die Maschinen-Elemente; ihre Berechnung und Konstruktion mit Rücksicht auf die neueren Versuche. 9te, vermehrte Auflage. 2 Bände. Stuttgart, Bergsträsser, 1903. 8vo. 22 + 848 pp., 59 plates. M. 32.00

BEHREND (B. A.). Le moteur d'induction. Traité succinct concernant sa théorie et son étude, avec de nombreux résultats d'essais et diagrammes. Paris, Loubat, 1903. 8vo. 2 + 117 pp.

——. See BERKITZ (P.).

BENISCHKE (G.). Die Grundgesetze der Wechselstromtechnik. Braunschweig, Vieweg, 1903. 8vo. 10 + 141 pp. M. 3.60

BERKITZ (P.). Induktionsmotoren. Ein Kompendium für Studirende und Ingenieure. Deutsche, autorisirte und erweiterte Bearbeitung von B. A. Behrend, The induction motor; unter Mitwirkung von W. Kübler herausgegeben. Berlin, Krayn, 1903. 8vo. 7 + 182 pp. M. 10.00

BIRCKENSTAEDT (M.). Verallgemeinerung der in den "Principien der Mechanik für mehrere unabhängige Variable von Herrn L. Koenigsberger aus Heidelberg" dargestellten Hülfsätze über das kinetische Potential. (Diss.) Heidelberg, 1902. 4to. 51 pp.

BIRVEN (H.). Das Fachwerk. Eine Einführung in die statische Berechnung desselben; zugleich ein Repetitorium für den ausübenden Techniker. Hildburghausen, 1903. 8vo. 4 + 24 pp. Cloth. M. 1.50

BODMER (G. R.). Hydraulic motors and turbines; for the use of engineers, manufacturers, and students. New York, Van Nostrand, 1902. 8vo. Cloth. $5.00

BOLTZMANN (L.). Ueber die Prinzipien der Mechanik. Zwei akademische Antrittsreden (1900 und 1902). Leipzig, Hirzel, 1903. 8vo. 48 pp. M. 1.00

BRINK (A.). Internal ballistics. Part I: Properties of powders and their action in closed vessels and in the bore of a gun. St. Petersburg, 1901–1902. 8vo. 12 + 370 pp. (Russian.) R. 3.40

CAIN (W.). Theory of steel-concrete arches and of vaulted structures; being a revised edition of "Voussoir arches applied to stone bridges, tunnels, culverts, and domes." New York, Van Nostrand, 1902. 16mo. Boards. (Van Nostrand's science series, No. 42.) $0.50

COMSTOCK (G. C.). A text-book of field astronomy for engineers. New York, Wiley, 1902. 8vo. 10 + 202 pp. Cloth. $2.50

FERRON (E.). Théorie mécanique des bicycles et locomotives. Paris, Gauthier-Villars, 1902. 4to. 19 pp., 4 plates. Fr. 2.00

——. Contribution à la théorie mathématique de la chaleur. Paris, Gauthier-Villars, 1902. 8vo. 45 pp., 1 plate. Fr. 2.00

GEUER. Die Genauigkeit geometrischer Zeichnungen behandelt nach dem Gaussschen Ausgleichungsverfahren, wonach die Summe der Fehlerquadrate ein Minimum wird. (Progr.) Durlach, 1902. 4to. 39 pp., 4 plates.

GOEDSEELS (P. J. S.). Théorie des erreurs d'observations. Paris, Gauthier-Villars, 1902. 8vo. 13 + 168 pp. Fr. 7.50

GREENE (C. E.). Graphics for engineers, architects, and builders; a manual for designers and a text-book for scientific schools. Part 3: Trusses and arches analyzed and discussed by graphical methods. 3d edition, revised. New York, Wiley, 1903. 12mo. 194 pp., folding diagrams. Cloth. $2.50

GUY (A. E.). Experiments on the flexure of beams; discovery of new laws; with diagrams and figures. New York, Van Nostrand, 1902. 8vo. Cloth. $1.25

HALL (W. S.). Descriptive geometry; text and plates. 2 volumes. New York, Van Nostrand, 1902. Text 8vo, plates 4to. Cloth. $3.50

HOFF (J. H. VAN'T). Vorlesungen über theoretische und physikalische Chemie. Heft 2: Die chemische Statik. 2te Auflage. Braunschweig, Vieweg, 1903. 8vo. 10 + 150 pp. M. 4.00

INTERNATIONAL CATALOGUE of scientific literature. First annual issue [for 1901]. B. Mechanics. Published for the International Council by the Royal Society of London. London, Harrison. Vol. IX: 1902 (October). 8vo. 14 + 128 pp.

JAMISON (A. P.). Elements of mechanical drawing; a course in applied descriptive geometry. [Lafayette, Ind., Jamison, 1902.] 8vo. 79 pp. $1.50

JANSSEN (A.). Cours de mécanique rationnelle. Bruxelles, 1902. 8vo. 352 pp. M. 9.00

JOHNSON (L. J.). Statics by algebraic and graphic method, intended primarily for students of engineering and architecture. New York, Wiley, 1903. 8vo. 8 + 133 pp. Cloth. $2.00

JONES (F. R.). Machine design. Part I: Kinematics of machinery. 3d edition, revised. New York, Wiley, 1903. 8vo. 6 + 163 pp. Cloth. $1.50

KALÄHNE (A.). Schallgeschwindigkeit und Verhältnis der spezifischen Wärmen der Luft bei hoher Temperatur. (Habilitationsschrift.) Heidelberg, 1902. 8vo. 34 pp.

KAYSER (H.). Die Elektronentheorie. Bonn, Röhrscheid & Ebbecke, 1903. 8vo. 32 pp. M. 0.80

KINSLEY (C.). A systematic method of calculating the dimensions of direct-current dynamo-electric machines. Chicago, University of Chicago Press, 1903. 4to. 14 pp. (University of Chicago decennial publications; printed from Vol. 9.) $0.25

LINEHAM (W. J.). Text-book of mechanical engineering. 6th edition, enlarged. London, Chapman, 1903. 8vo. 1066 pp. Cloth. 12s. 6d.

LONEY (S. L.). Solutions of the examples in the elements of hydrostatics. London, Clay, 1903. 12mo. 146 pp. Cloth. 5s.

MANN (C. R.). Manual of advanced optics. Chicago, Scott, Foresman & Co., 1902. 12mo. 196 pp. Cloth. $2.00

MAURER (E. R.). Technical Mechanics. Part I. New York, Wiley, 1903. 8vo. 10 + 158 pp. Cloth. $2.00

MÜLLER (J.). Ueber Schallgeschwindigkeit in Röhren. (Diss.) Bonn, 1903. 8vo. 48 pp.

PERRIN (J.). Traité de chimie physique. Les principes. Paris, Gauthier-Villars, 1903. 8vo. 26 + 300 pp. Fr. 10.00

RAYLEIGH. See STRUTT (J. W.).

REYNOLDS (O.). Papers on mechanical and physical subjects. Vol. 3: Sub-mechanics of the universe. London, Clay, 1903. 8vo. 272 pp. Cloth. 10s. 6d.

RODET (J.). Distribution de l'énergie par courants polyphasés. 2e édition, entièrement refondue. Paris, Gauthier-Villars, 1903. 8vo. 9 + 561 pp. Fr. 15.00

SCHILLER (N.). The fundamental laws of thermodynamics. Kiev, 1903. 8vo. 51 pp. (Russian.)

——. On the possibility of establishing the mechanics of masses without using the auxiliary notion of force. Kiev, 1902. 8vo. 8 pp. (Russian.)

SEVIN (O.). Beiträg zur Kenntnis von sieben- und mehrgliedrigen heteroatomigen Ringen. (Diss.) Heidelberg, 1902. 8vo. 54 pp.

SEWELL (T.). The elements of electrical engineering; a first year's course for students. New York, Van Nostrand, 1903. 8vo. 332 pp. Cloth. $3.00

STRUTT (J. W.) [Baron Rayleigh]. Scientific Papers. Vol. 4: 1892–1901. London and New York, Macmillan, 1903. 8vo. 14 + 604 pp., portrait. Cloth. $5.00

VONDERLINN (J.). Lehrbuch des Projektionszeichnens. Teil 4, 1ste Hälfte; Ebene und Raumkurven; abwickelbare Flächen; die Kugelfläche. Nach System Kleyer. Bremerhaven, Vangerow, 1903. 8vo. 11 + 252 pp. (Kleyer's Encyklopädie der gesamten mathematischen, technischen und exakten Naturwissenschaften.) M. 6.00

WEYHER (C. L.). Sur l'éther. Paris, Gauthier-Villars, 1903. 8vo. 12 pp. Fr. 0.75